KTH Matematik

Solution to Homework 2
Mathematical Systems Theory, SF2832
Fall 2013
For reference only.

1. (*a*) Consider a time-invariant system

$$\dot{x} = Ax,$$

where $x \in R^n$, A is nilpotent ($A^k = 0$ for some k). For what nilpotent A is $x = 0$ (critically) stable? . . . . (1p)

Answer: $A = 0$.

(*b*) Consider

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

1. For what a_1 and a_2 the Lyapunov equation $A^T P + PA + C^T C = 0$ has a positive definite solution?

Answer: $a_1 \neq a_2$ and both are negative.

2. Find the positive definite solution when it exists . . . . (2p)

Answer: omitted.

2. Given the following system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u \quad (1)$$

$$y = cx = \begin{bmatrix} c_1 & c_2 \cdots & c_n \end{bmatrix} x, \quad (2)$$

(*a*) If $x(0) = 0$ and $u = \sin(t)$, discuss necessary and sufficient conditions on the coefficients of c such that through observing $y(t)$ we can always draw the correct conclusion on if A defined in (1) is a stable matrix. . . . . (2p)

Answer: The necessary and sufficient condition is that all modes associated with the unstable eigenvalues of A should not be missed in $y(t)$, namely there should not be any cancellation of poles with nonnegative real-part by zeros.

(*b*) Suppose $c_n = 1$ and $y(0) = 0$. Find a feedback control $u = Kx$ that forces $y(t) = 0$ $\forall t \geq 0$ in the closed-loop system. . . . . (1p)

Answer: $y(t) = 0$ $\forall t \geq 0$ implies $\dot{y} = 0$ $\forall t \geq 0$, i.e. $\dot{y} = c\dot{x} = c_1 x_2 + \cdots + c_{n-1} x_n - \sum a_i x_i + u = 0$, which gives the control.

(c) Discuss necessary and sufficient conditions on $c_1, \cdots, c_{n-1}$ such that in the closed-loop system we obtain in 2.b, $y(t) = 0,\ \forall t \geq 0$ implies that $x(t) \to 0$ as $t \to \infty$. .... (2p)
Answer: All the zeros are with negative real-parts.

3. Consider

$$R(s) = \begin{bmatrix} \frac{1}{s(s+2)} & \frac{1}{s+2} \\ \frac{k}{s(s+2)} & \frac{1}{s+2} \end{bmatrix},$$

where k is a constant.

(a) Determine the standard observable realization of $R(s)$. .... (2p)
Answer: omitted.

(b) What is the McMillan degree of $R(s)$? .... (2p)
Answer: $\delta R = 3$ if $k \neq 1$, $\delta R = 2$ otherwise.

(b) Let $k = 1$, derive a minimal realization. .... (2p)
Answer: omitted.

4. This problem is about Lyapunov equations. Consider

$$\dot{x} = Ax + Bu.$$

Assume (A, B) is controllable. Let $u = -B^T V^{-1}(0, t_1)x$, where $V(0, t_1) = \int_0^{t_1} e^{-At} BB^T e^{-A^T t} dt$.

(a) Show $\forall t_1 > 0$, $\bar{A} = A - BB^T V^{-1}(0, t_1)$ is a stable matrix. .... (2p)
Answer: Clearly $V(0, t_1) > 0$. Since $\frac{d}{dt} e^{-At} BB^T e^{-A^T t} = -Ae^{-At} BB^T e^{-A^T t} - e^{-At} BB^T e^{-A^T t} A^T$, $AV(0, t_1) + V(0, t_1)A^T = BB^T - e^{-At_1} BB^T e^{-A^T t_1}$. Then $(A - BB^T V^{-1})V + V((A - BB^T V^{-1})^T = -BB^T - e^{-At_1} BB^T e^{-A^T t_1} \leq -BB^T$, thus $A - BB^T V^{-1}$ is a stable matrix.

(b) Show $\lim_{t_1 \to \infty} \bar{A} = A$ if A self is a stable matrix. .... (2p)
Answer: It is easy to show that $V(0, \infty)^{-1} = 0$.

(c) What is $\lim_{t_1 \to \infty} \bar{A}$ if $-A$ is a stable matrix? .... (2p)
Answer: In this case we have $(A - BB^T V^{-1})V + V((A - BB^T V^{-1})^T = -BB^T$. Thus $A - BB^T V^{-1} = (-BB^T - V(A - BB^T V^{-1})^T)V^{-1} = -VAV^{-1}$.