KTH Matematik

Solution to Homework 3
Mathematical Systems Theory, SF2832
Fall 2013
For reference only.

1. Determine a state feedback K such that the eigenvalues of the closed-loop system $\dot{x} = (A + BK)x$ are located in $\{-1, -1, -2, -2\}$, for the case when

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

. . . . (3p)

Answer: omitted.

2. Consider a state space realization (A, b, c) as follows

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} c_1 & c_2 & 1 \end{bmatrix} x,$$

where $c_1,\ c_2$ are constants.

(a) For what $c_1,\ c_2$ is the system observable? . . . . (1p)

Answer: $c_1 \neq 0$.

(b) Design a feedback controller $u = kx$ such that dimension of the unobservable subspace for $(c, A + bk)$ is maximized while $A + bk$ has at least one eigenvalue at -1. . . . . (3p)

Answer: To maximize the unobservable subspace is to have as many pole/zero cancellations as possible. Thus the characteristic polynomial for $A + bk$ should be $(s^2 + c_2 s + c_1)(s + 1)$ and we obtain the k accordingly.

(c) Assume now that $c_1 = 2,\ c_2 = 3$ and the full state is not available. Can we always design an observer-based control that stabilizes the overall system, with the closed-loop poles located at $\{-1, -2, -3\}$ and the observer dynamics having poles at $\{-1, -2, -3\}$? . . . . (2p)

Answer: Yes, since the system is both controllable and observable.

3. Consider a state space system (A,b) as follows

$$\dot{x}_1 = x_1 + ax_2$$
$$\dot{x}_2 = ax_1 + u,$$

where a is a constant, and the cost function

$$J = \int_0^{t_1} (x_2^2 + \epsilon^2 u^2) dt.$$

Assume $x^*(t)$ is the optimal trajectory for a given initial point $(x_1(0)\ x_2(0))^T$ with the optimal control $u = -\epsilon^{-2} b^T P(t) x(t)$.

(*a*) For what a is $P(t)$ positive definite $\forall t < t_1$?. .... (2p)
Answer: $a \neq 0$.

(*b*) Now let $t_1 = \infty$. For the case where P_∞ is positive definite, compute the eigenvalues of $A - \epsilon^{-2} bb^T P_\infty$ as $\epsilon \to 0$. .... (3p)
Answer: $-1, -\infty$.

4. Consider a one-dimensional system

$$x(t+1) = ax(t) + v(t)$$
$$y(t) = x(t) + w(t),$$

where $a \neq 0,\ v, w$ are uncorrelated white noises, with covariances $\sigma,\ r$ respectively.

Comment: This problem is given to show that in general it is difficult to solve explicitly a Kalman filter (this is just a scalar system). On the other hand, we will be generous in grading.

(*a*) Design a Kalman filter $\hat{x}(t)$ for $x(t)$. .... (2p)
Answer: omitted.

(*b*) Express the covariance matrix $p(t) = E\{(x(t) - \hat{x}(t))^2\}$ in terms of $a, \sigma,\ r$.(2p)
Answer: $p(t+1) = \frac{a^2 r p(t)}{r + p(t)} + \sigma$, but this is difficult to solve. Let $p_0 = \frac{a^2 r p_0}{r + p_0} + \sigma$, which gives a positive solution $p_0 = \frac{1}{2}(a^2 r + \sigma - r + \sqrt{(a^2 r + \sigma - r)^2 + 4\sigma r})$. Let $p(t) = \delta p(t) + p_0$, then we have (after straight forward manipulation) $\frac{1}{\delta p(t+1)} = \frac{p_0 + r}{a^2 r + \sigma - p_0} \frac{1}{\delta p(t)} + \frac{1}{a^2 r + \sigma - p_0}$, which is a linear system thus can be solved.

(*c*) What is $a - ak(t)$ as $t \to \infty$ (where $k(t)$ is the Kalman gain)? .... (2p)
Answer: Since $|\frac{p_0 + r}{a^2 r + \sigma - p_0}| > 1$, $\frac{1}{\delta p(t)}$ diverges thus $p(t)$ converges to p_0. Since $k(t) = \frac{p}{p+r}$, $|a(1 - \frac{p_0}{p_0 + r})| = |\frac{ar}{p_0 + r}| < |\frac{2ar}{a^2 r + \sigma + r}| \leq \sqrt{\frac{r}{r + \sigma}} < 1$, which shows the Kalman filter converges.