KTH Matematik

Solution to Homework 1
Mathematical Systems Theory, SF2832
Fall 2014
For reference only.

1. Find the state transition matrix $\Phi(t,s)$ for the following systems

(a) $\dot{x}(t) = \begin{bmatrix} 1 & \sin(t) \\ 0 & 1 \end{bmatrix} x(t)$

. . . . (2p)

Answer: $\Phi(t,s) = \begin{bmatrix} e^t & e^t(\cos(s) - \cos(t)) \\ 0 & e^t \end{bmatrix}$.

(b) $\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix} x(t)$.

. . . . (3p)

Answer: $det\ (sI - A) = (s+1)(s^2+1)$. Then we compute $e^{At} = \mathcal{L}^{-1}[(sI-A)^{-1}]$. The detail is omitted.

2. (a) Let

$$\dot{x} = A(t)x$$

and assume $A^T(t) = -A(t)$. Show $\Phi^T(t,s) = \Phi^{-1}(t,s)$. . . . . (2p)

Answer: Since $\frac{d}{ds}\Phi(t,s) = -\Phi(t,s)A(s)$, $\frac{d}{ds}\Phi^T(t,s) = A(s)\Phi^T(t,s)$, Thus, $\Phi^T(t,s) = \Phi(s,t) = \Phi^{-1}(t,s)$.

(b) Let

$$\dot{x} = A(t)x.$$

Show that if $\int_s^t A(\tau)d\tau$ and $A(t)$ commute for all t, s, then the state transition matrix $\Phi(t,s) = exp(\int_s^t A(\tau)d\tau)$. . . . . (3p)

Answer: We use Taylor expansion to show that $\frac{d}{dt}exp(\int_s^t A(\tau)d\tau) = A(t)exp(\int_s^t A(\tau)d\tau)$. For this purpose we only need to show $\frac{d}{dt}(\int_s^t A(\tau))^k = (\frac{d}{dt}(\int_s^t A(\tau))(\int_s^t A(\tau))^{k-1} + \cdots + (\int_s^t A(\tau))^{k-1}\frac{d}{dt}(\int_s^t A(\tau)) = kA(t)(\int_s^t A(\tau))^{k-1}$.

3. Consider

$$\begin{aligned} \dot{x} &= Ax,\ x \in R^n \\ y &= Cx,\ y \in R^p \\ x(0) &= x_0 \end{aligned}$$

where A and C are constant matrices.

(a) Show that if $x(0) \in ker\ \Omega$, then $x(t) \in ker\ \Omega$, $\forall t \geq 0$, where $\Omega = (C^T, A^TC^T, \cdots, (A^{n-1})^TC^T)^T$. (3p)

Answer: If $\Omega x(0) = 0$, then by Cayley-Hamilton theorem, $\Omega Ax(0) = 0$, thus $Ax(0) \in ker\ \Omega$. We then show $A^k x(0) \in ker\ \Omega$ by repeating this step, thus $e^{At}x(0) \in ker\ \Omega$.

(b) Show that the above system is observable if and only if the only solution that satisfies $Cx(t) = 0, \forall t \geq 0$ is $x(t) = 0$. . . . . (2p)

Answer: $Cx(t) = 0, \forall t \geq 0$ implies that for any k, $Cx^{\{k\}}(t) = CA^k x(t) = 0, \forall t \geq 0$. Once again by Cayley-Hamilton we have $x(t) \in ker\ \Omega$ and the conclusion follows.

4. The following is linearized model of a so-called inverted double pendulum

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & 0 & -a_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & -a_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3a_3 & 0 & -a_4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ b_1 \\ 0 \\ -15b_2 \\ 0 \\ -b_2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and all the parameters are positive.

(a) Check controllability for this system. Can we find at least one set of parameters a_i, b_i such that the system is controllable? (you can use, for example, Maple to help). . . . . (3p)

Answer: The purpose here is to let you practice Maple. As we will see, for almost all parameters the system is controllable.

(b) Is the system observable? . . . . (2p)

Answer: One can easily see observability by using 3(b).