KTH Matematik

Homework 1
Mathematical Systems Theory, SF2832
Fall 2015
You may use min(5,(your score)/4) as bonus credit on the exam.

(Your homework should be handed in to Yuecheng Yang before the deadline)

1. Find the state transition matrix $\Phi(t,s)$ for the following systems

(a) $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} x(t)$

.... (2p)

solution: omitted.

(b) $\dot{x}(t) = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix} x(t).$

.... (3p)

solution: Denote $\omega = \frac{1}{\sqrt{14}}(3\ 2\ 1)^T$, $\theta = \sqrt{14}t$,

$$e^{At} = \begin{pmatrix} \omega_1^2 + (1-\omega_1^2)\cos\theta & \omega_1\omega_2(1-\cos\theta) - \omega_3\sin\theta & \omega_1\omega_3(1-\cos\theta) + \omega_2\sin\theta \\ \omega_1\omega_2(1-\cos\theta) + \omega_3\sin\theta & \omega_2^2 + (1-\omega_2^2)\cos\theta & \omega_2\omega_3(1-\cos\theta) - \omega_1\sin\theta \\ \omega_1\omega_3(1-\cos\theta) - \omega_2\sin\theta & \omega_2\omega_3(1-\cos\theta) + \omega_1\sin\theta & \omega_3^2 + (1-\omega_3^2)\cos\theta \end{pmatrix}$$

2. (a) Suppose an $n \times n$ matrix $A(t)$ satisfies $\dot{A} = KA - AK$, $A(0) = A_0$, where K is a constant $n \times n$ matrix. Show

$$A(t) = e^{Kt}A_0e^{-Kt},$$

and the eigenvalues of A are independent of t. .... (2p)

solution: By differentiating $e^{Kt}A_0e^{-Kt}$ we show the identity easily. Since $A(t)$ has the same eigenvalues as A_0, they are independent of t.

(b) For the same A in (a), consider

$$\dot{x} = A(t)x.$$

Show that a fundamental matrix is $\Psi(t) = e^{Kt}e^{(A_0-K)t}$. .... (3p)

solution: Using $A(t) = e^{Kt}A_0e^{-Kt}$, we can verify that $\dot{\Psi} = A(t)\Psi$.

3. Consider

$$\dot{x} = Ax + bu,$$

where

$$A = \begin{bmatrix} 0 & -a_1 & a_2 \\ a_1 & 0 & -a_3 \\ -a_2 & a_3 & 0 \end{bmatrix}, \ b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

(a) Show that $\forall x(0),\ \|e^{At}x(0)\| = \|x(0)\|$. .... (2p)
solution: Note that $A^T = -A$. $\|x(t)\|^2 = x^T(0)e^{A^Tt}e^{At}x(0) = x^T(0)x(0) = \|x(0)\|^2$.

(b) Show that if $\|a\| \neq 0$, then there exists b such that (A, b) is controllable, where $a = (a_1\ a_2\ a_3)^T$. .... (3p)
solution: Let $\bar{a} = (a_3\ a_2\ a_1)^T$, then $Ab = \bar{a} \times b,\ A^2b = \bar{a} \times (\bar{a} \times b)$. As long as b is not orthogonal to a or $\bar{a}$, (b, Ab, A^2b) has full rank.

4. The following is linearized model of an inverted pendulum

$$\dot{x} = Ax + Bu$$

where g is the acceleration of gravity, and

$$A = \begin{bmatrix} 0 & 1 \\ g & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

(a) Check controllability for this system. Can we find control $u = k_1x_1 + k_2x_2$ such that the closed-loop system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ g - k_1 & -k_2 \end{bmatrix} x$$

is asymptotically stable? .... (3p)
solution: Obviously yes.

(b) Can we find a scalar output $y = cx$ such that the system is observable? .. (2p)
solution: Obviously yes.