KTH Matematik

Solution to Homework 2
Mathematical Systems Theory, SF2832
Fall 2015
You may use min(5,(your score)/5) as bonus credit on the exam.

1. Consider the pair (C, A), where

$$A = \begin{bmatrix} 0 & a_1 \\ 1 & a_2 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

For what a_1 and a_2 the Lyapunov equation $A^T P + PA + C^T C = 0$ has

(a) a positive definite solution? .... (1p)
Answer: $a_1 < 0,\ a_2 < 0$.

(b) a negative definite solution (-P is positive definite)? .... (1p)
Answer: $a_1 < 0,\ a_2 > 0,\ a_1 + a_2 \neq 1$

(c) Find P for $a_1 = -1,\ a_2 = -2$ and verify that $P > 0$. .... (2p)
Answer: omitted.

2. You will in this problem derive and investigate a number of realizations for the transfer function

$$R(s) = \begin{bmatrix} \dfrac{\gamma}{s+1} & \dfrac{1}{s+1} \\ \dfrac{1}{(s+1)(s+3)} & \dfrac{1}{(s+1)(s+3)} \end{bmatrix},$$

where $\gamma > 0$ is a constant.

(a) Determine the standard reachable realization of $R(s)$. .... (1p)
Answer: omitted.

(b) What is the McMillan degree of $R(s)$? .... (2p)
Answer: $\delta(R) = 3$ if $\gamma \neq 1$, otherwise $\delta(R) = 2$.

(c) Find a minimal realization of $R(s)$ for $\gamma \neq 1$, and verify that your solution (A, B, C) satisfies $C(sI - A)^{-1}B = R(s)$. .... (3p)
Answer: There are several ways to derive a minimal realization. For example, $\dot{x}_1 = -3x_1 + x_2 + x_3, \dot{x}_2 = -x_2 + u_1, \dot{x}_3 = -x_3 + u_2, y_1 = \gamma x_2 + x_3, y_2 = x_1$.

3. (a) Given the control system

$$\dot{x} = Ax + bu$$
$$A = diag\{a_1, \cdots, a_n\}, b = (b_1, \cdots, b_n)^T,$$

where all the coefficients are real. What is the necessary and sufficient condition on the diagonal elements of A such that the pole placement problem is solvable for some choices of b? . . . . (2p)

Answer: They must be distinct.

(*b*) Consider a controllable and observable system

$$\dot{x} = Ax + bu$$
$$y = cx,$$

where, $x \in R^n$, $u \in R$, $y \in R$. We let the transfer function $r(s) = c(sI - A)^{-1}b = \frac{n(s)}{d(s)}$. Show that a feedback control $u = kx = \sum_{i=1}^{n} k_i x_i$ makes $(c, A + bk)$ unobservable if and only if $A+bk$ has an eigenvalue λ_0 that satisfies $n(\lambda_0) = 0$. . . . . (3p)

Answer: Since $\dot{x} = (A+bk)x+bv$ is always controllable, the closed-loop system becomes unobservable iff there is a zero-pole cancellation.

(*c*) Consider

$$\dot{x}_1 = Px_1 + qx_2$$
$$\dot{x}_2 = q^T x_1 + u$$
$$y = x_2,$$

where $x_1 \in R^n, x_2 \in R$, and P is symmetric. Show that one can use high gain output feedback $u = -ky,\ k > 0$ to stabilize the system if P is a stable matrix, namely, in this case

$$\begin{pmatrix} P & q \\ q^T & -k \end{pmatrix}$$

is a stable matrix for sufficiently large k. (Hint: Google and use Schur Complement Lemma). . . . . (5p)

Answer: Since

$$\begin{pmatrix} P & q \\ q^T & -k \end{pmatrix}$$

is symmetric, to show it is stable is to show it is negative definite when k is sufficient large. By Schur Complement Lemma, we only need to show $k+q^T P^{-1} q$ is positive when k is sufficiently large.

4. Suppose the following is a realization of a given $R(s)$:

$$(A, B, C) = \left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \right)$$

(a) Design a feedback control $u = Kx$ that assigns poles to $\{-1, -1, -1, -2\}$. (2p)

(b) Is the realization minimal? If not, use Kalman decomposition to find a minimal realization. . . . . (3p)

Answer: omitted.