KTH Matematik

Solution to Homework 3
Mathematical Systems Theory, SF2832
Fall 2015
You may use min(5,(your score)/4) as bonus credit on the exam.

1. Consider a state space realization (A, b, c) as follows

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} w$$
$$y = \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} x,$$

where $w(t)$ is an unknown disturbance signal.

(a) Can we find $u = kx$ such that in the closed-loop system $y(t)$ is not affected by $w(t)$ (i.e. $y(t)$ is determined only by the initial state)? .... (3p)
Answer: This is equivalent to finding k such that $\int_0^t ce^{A(t-s)}(1 \ -1 \ 1)^T w(s)ds = 0$, which implies $(1 \ -1 \ 1)^T$ should be in the unobservable subspace of $(c, A+bk)$. Since $c(1 \ -1 \ 1)^T \neq 0$, no k will make this happen.

(c) Assume now that $w = 0$ and the full state is not available. Can we design an observer-based control that stabilizes the overall system, with the closed-loop poles located at $\{-1, -1, -1\}$ and the observer dynamics having poles at $\{-1, -1, -1\}$? .... (2p)
Answer: omitted.

2. Consider a state space system as follows

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 + u$$
$$y = x_2.$$

Let $P(t)$ denote the solution to the dynamical Riccati equation associated with the following optimal control problem

$$min \ J = \int_0^{t_1} (y^2 + u^2)dt.$$

(a) Show that $P(t)$ is positive definite $\forall t < t_1$. .... (2p)
Answer: This is true since the system is observable.

(b) Compute $\lim_{t_1 \to \infty} P(0)$, i.e., the limiting value of $P(t)$ at $t = 0$. .... (3p)
Answer: $p_1 = p_3 = \sqrt{5}, p_2 = 2$.

3. Consider the algebraic Riccati equation

$$A^T P + PA - PBB^T P + C^T C = 0.$$

(a) Assume P is a real **positive semidefinite** solution. Show that if (C, A) is observable, then P is positive definite. .... (2p)

Answer: Let $x \in Ker\ P$. By manipulating the ARE, we can show $x \in Ker\ C$ and $Ker\ P$ is A-invariant. Since (C, A) is observable, the A-invariant subpspace contained in $Ker\ C$ ($Ker\ \Omega$) is 0. Thus $Ker\ P = 0$

(b) Assume P is a real **positive semidefinite** solution. If (A, B) is controllable, can we draw the conclusion that P is positive definite? .... (3p)

Answer: No. For example, $C = 0$ and A is a stable matrix, then $P = 0$.

4. Consider a one-dimensional system

$$\begin{aligned} x(t+1) &= ax(t) + v(t) \\ y(t) &= x(t) + w(t), \end{aligned}$$

where $a \neq 0,\ v, w$ are uncorrelated white noises, with covariances $\sigma,\ r$ respectively.

(a) Design a Kalman filter $\hat{x}(t)$ for $x(t)$. .... (1p)

(b) Express the covariance matrix $p(t) = E\{(x(t) - \hat{x}(t))^2\}$ in terms of $a, \sigma,\ r$.(2p)

(c) What is $a - ak(t)$ as $t \to \infty$ (where $k(t)$ is the Kalman gain)? .... (2p)

Answer: omitted.