KTH Matematik

Solution to Homework 1
Mathematical Systems Theory, SF2832
Fall 2016
You may use min(5,(your score)/4) as bonus credit on the exam.

1. (*a*) Find the input-output description for $\ddot{y} = -y + u$,
where $y \in R$ is the output and $u \in R$ is the input. ....(2p)

Answer: A state space model for the system is: $\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1 + u,\ y = x_1$. Then

$$G(t) = [1\ 0]exp[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t][0\ 1]^T = [1\ 0]\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}[0\ 1]^T = \sin t.$$

(*b*) Find the state transition matrix for the following system

$$\dot{x}(t) = \begin{bmatrix} 0 & t & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} x(t)$$

.... (2p)

Answer: Using the result from (a), we obtain first the state transition matrix for the subsystem consisting of x_2 and x_3.
Then we have $\dot{x}_1 = tx_2(t) = t\cos t x_2(0) + t \sin t x_3(0)$, thus

$$x_1(t) = x_1(0) + (\cos t + t \sin t)x_2(0) + (\sin t - t\cos t)x_3(0).$$

We can then easily write down the state transition matrix for the whole system. The rest is omitted.

2. Consider the rotational motion of a point x in R^3 with respect to the origin:

$$\dot{x} = \omega \times x,$$

where $\omega = (\omega_1, \omega_2, \omega_3)^T \neq 0$, a constant vector, is the angular velocity, and "×" is the vector cross product.

(*a*) Express the kinematics of $x(t)$ in the form of $\dot{x} = Ax$..................... (1p)

Answer: $A = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$

(*b*) Compute e^{At}. .... (4p)

Answer: Let $|\omega| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2}$,

$$e^{At} = e^{\frac{A}{|\omega|}|\omega|t} = I + \frac{A}{|\omega|}\sin(|\omega|t) + \frac{A^2}{|\omega|^2}(1 - \cos(|\omega|t)).$$

3. Consider

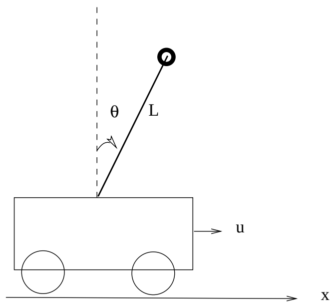
$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

where $x \in R^n,\ u \in R^m,\ y \in R^p$, A, B and C are constant matrices. Let Ω be the observability matrix as is defined in the compendium. State and prove the necessary and sufficient condition such that for any $x(0) \in ker\ \Omega$, $x(t) \in ker\ \Omega$, $\forall t \geq 0$, no matter what $u(t)$ is used.

. . . . (3p)

Answer: We show the condition $\mathcal{R} \subseteq ker\ \Omega$ is necessary and sufficient. Sufficiency is obvious. We use contradiction to show the necessity. Assume $x_1 \in \mathcal{R}$, but not in $ker\ \Omega$. Since the origin is in both $\mathcal{R}$ and $ker\ \Omega$, there is a control that makes $x(T) = x_1$ starting from the origin, where T is any positive number. This draws a contradiction.

4. Consider the inverted pendulum as we did in the lecture



Figur 1: Inverted pendulum on a cart.

The following equation describes the motion of the pendulum around the equilibrium $\theta = 0$:

$$L\ddot{\theta} - g\sin(\theta) + \ddot{x}\cos(\theta) = 0$$

(a) We consider $\ddot{x}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the linearized system (i.e. let $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$). . . . . (1p)

(b) Show the model you derive in (a) is both controllable and observable. . . . (2p)

Answer: Omitted.

5. Now consider an inverted pendulum with oscillatory base

The following equation describes the motion of the pendulum around the equilibrium $\theta = 0$:

$$L\ddot{\theta} - g\sin(\theta) - \ddot{z}\sin(\theta) = 0$$

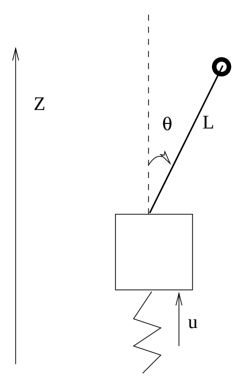
(*a*) We consider $\ddot{z}$ as the input u and θ as the output y. Let $x_1 = \theta$ and $x_2 = \dot{\theta}$. Derive the state space model for the *linearized* system (i.e. let $\sin(\theta) \approx \theta$).(1p)

Answer: $\dot{x}_1 = x_2,\ \dot{x}_2 = \frac{g}{L}x_1$.

(*b*) Is the model you derive in (a) controllable? .... (1p)

Answer: Obviously not.

(*c*) Let $z = 0.1\sin(\omega t)$ (be careful with the variable!). Use Matlab simulation to find out what will happen to the motion (use the original nonlinear model!) of the pendulum near $\theta = 0$ when the oscillation is "slow" and when the oscillation is "fast". Take $L = 1,\ \theta(0) = 0.02$ and $\dot{\theta}(0) = 0$. ....(3p)



Figur 2: Pendulum with oscillatory base.