KTH Matematik

Solution to Homework 3
Mathematical Systems Theory, SF2832
Fall 2016
You may use min(5,(your score)/4) as bonus credit on the exam.

1. Consider

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

(a) Find the values of $k = (k_1\ k_2)$ such that $A + bk$ has two identical eigenvalues. (2p)
Answer: $k_2^2 + 4k_1 = 0$.

(b) Find the values of h such that $A + bhc$ has two identical eigenvalues...... (1p)
Answer: $h = 0$.

(c) Can we conclude that for almost all k and h, the two matrices have distinct eigenvalues? .... (1p)
Answer: Yes.

2. Consider

$$\dot{x} = Ax + Bu$$
$$x(0) = x_0,$$

where (A, B) is controllable. Given the cost function

$$J(u, t_1) = \int_0^{t_1} u^T u dt,$$

with $x(t_1) = 0$, we know from Chapter 3 that $u^* = -B^T e^{A^T(t_1-t)} W(0, t_1)^{-1} e^{At_1} x_0$ is the optimal control.

(a) Show that u^* can be rewritten as $u^* = -B^T e^{A^T(t_1-t)} W(t, t_1)^{-1} e^{A(t_1-t)} x(t)$ (You are not allowed to argue by Bellman's Principle here). .... (3p)
Answer: With the optimal control, we have

$$x(t) = e^{At} x_0 - \int_0^t e^{A(t-s)} B B^T e^{A^T(t_1-s)} W(0, t_1)^{-1} e^{At_1} x_0 ds.$$

Plug in $x(t)$ and use the fact $W(0, t_1) = \int_0^t e^{A(t_1-s)} B B^T e^{A^T(t_1-s)} ds + W(t, t_1)$, thus $W(0, t_1)^{-1} = W(t, t_1)^{-1} - W(t, t_1)^{-1} \int_0^t e^{A(t_1-s)} B B^T e^{A^T(t_1-s)} ds W(0, t_1)^{-1}$, we can draw the conclusion.

(b) Show that $J(u^*, t_2) < J(u^*, t_1)$ if $t_2 > t_1$. .... (3p)
Answer: We can easily calculate that $J(u^*) = x_0^T e^{A^T t_1} W(0,t_1)^{-1} e^{At_1} x_0 := x_0^T P(t_1) x_0$. Since $P(t_1)^{-1} = \int_0^{t_1} e^{-As} BB^T e^{-A^T s}$, we have $P^{-1}(t_2) - P^{-1}(t_1) > 0$ if $t_2 > t_1$. Thus $P(t_1) - P(t_2) > 0$.

3. Consider

$$\dot{x}_1 = ax_1 + x_2$$
$$\dot{x}_2 = x_1 + u,$$

where a is a constant, and the cost function

$$J = \int_0^\infty (x_2^2 + \epsilon^2 u^2) dt.$$

As $\epsilon \to 0$, the optimal control problem becomes the so-called "cheap control" problem.

Assume $x^*(t)$ is the optimal trajectory for a given initial point $(x_1(0)\ x_2(0))^T$.

(a) For $a > 0$, find $\lim_{\epsilon \to 0} x_1^*(t)$. .... (2p)
Answer: In the corresponding ARE, let $p_2 = \epsilon \bar{p}_2$ (where p_2 denotes the off-diagonal element), $p_3 = \epsilon \bar{p}_3$. Then as $\epsilon \to 0$ the ARE becomes $2ap_1 - \bar{p}_2^2 = 0,\ p_1 - \bar{p}_2\bar{p}_3 = 0,\ \bar{p}_3^2 - 1 = 0$.
When $a > 0$, $\bar{p}_2 = 2a$, then as $\epsilon \to 0$, $\dot{x}_1^*(t) = -ax_1^*(t)$.

(b) For $a < 0$, find $\lim_{\epsilon \to 0} x_1^*(t)$. .... (2p)
Answer: When $a < 0$, $\bar{p}_2 = 0$, then as $\epsilon \to 0$, $\dot{x}_1^*(t) = ax_1^*(t)$.

(c) Explain what happens to $\lim_{\epsilon \to 0} x_1^*(t)$ when $a = 0$. .... (2p)
Answer: Not well posed in this case.

4. (a) Suppose (A, B) is controllable and (C, A) is observable, and P is the positive definite solution to

$$AP + PA^T - PC^T(DRD^T)^{-1}CP + B^TB = 0.$$

Show $A - PC^T(DRD^T)^{-1}C$ is a stable matrix. .... (2p)
Answer: One could use the same reasoning as on p. 77 of the compendium.

(b) Consider

$$x(t+1) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} \frac{a^2}{2} \\ a \end{bmatrix} v(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + w(t),$$

where v, w are uncorrelated white noises, with covariances $q,\ r$ respectively.
Design Kalman filter for the system. .... (2p)
Answer: omitted.