SYSTEMTEORI - ÖVNING 2

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1. Summary on how to check reachability and observability

Consider the linear time-varying system

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + B(t)u(t), \quad x(t) \in \mathbb{R}^n \\ y(t) & = & C(t)x(t) + D(t)u(t). \end{array} \right. \tag{1.1}$$

For this system, we form reachability and observability Gramians:

$$W(t_0, t_1) := \int_{t_0}^{t_1} \Phi(t_1, s)B(s)B^T(s)\Phi^T(t_1, s)ds \tag{1.2}$$

$$M(t_0, t_1) := \int_{t_0}^{t_1} \Phi^T(t_1, s)C^T(s)C(s)\Phi(t_1, s)ds, \tag{1.3}$$

where $\Phi(t, s)$ is the transition matrix of the system.

Reachability. The state transfer from $x(t_0) = x_0$ to $x(t_1) = x_1$ is possible if and only if

$$x_1 - \Phi(t_1, t_0)x_0 \in \text{Im}W(t_0, t_1). \tag{1.4}$$

In particular, for time-invariant systems, let $\mathcal{R}$ be the **reachable subspace** defined by

$$\mathcal{R} := \text{Im}\left[B, AB, \cdots, A^{n-1}B\right]. \tag{1.5}$$

Then, the state transfer from any $x_0 \in \mathcal{R}$ to any $x_1 \in \mathcal{R}$ is possible in any time $\epsilon > 0$.

Observability. For a given input, the initial states $x(t_0) = a$ and $x(t_0) = b$ produce the same output on $[t_0, t_1]$ if and only if

$$a - b \in \ker M(t_0, t_1). \tag{1.6}$$

In particular, in time-invariant cases, for any t_0, t_1 $(t_0 < t_1)$,

$$\ker M(t_0, t_1) = \ker \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \tag{1.7}$$

which is called the **unobservable subspace**.

2. Reachability examples

Exercise 2.1 (LTV case). Let us consider the following LTV system:

$$\dot{x}(t) = \begin{pmatrix} 0 & 0 \\ t & 1/t \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) = A(t)x(t) + Bu(t), \quad t > 0.$$

Given the initial state $x(t_0) = x_0$, which states $x(t_1)$ are reachable?

A state $x(t_1)$ is reachable if and only if

$$x(t_1) - \Phi(t_1, t_0)x_0 \in \mathrm{Im}W(t_0, t_1)$$

where $W(t_0, t_1)$ is the reachability Gramian defined as:

$$W(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, s)B(s)B^T(s)\Phi^T(t_1, s)ds \tag{2.1}$$

In the last example of Övning 1, we obtained, for this system, the transition matrix $\Phi(t, s)$ as

$$\Phi(t, s) = \begin{pmatrix} 1 & 0 \\ t(t-s) & t/s \end{pmatrix}.$$

So, the Gramian is[1]:

$$\begin{aligned} W(t_0, t_1) &= \int_{t_0}^{t_1} \begin{pmatrix} 1 & 0 \\ t_1(t_1 - s) & t_1/s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & t_1(t_1 - s) \\ 0 & t_1/s \end{pmatrix} ds \\ &= (t_1 - t_0) \begin{pmatrix} 1 & t_1 \frac{(t_1 - t_0)}{2} \\ t_1 \frac{(t_1 - t_0)}{2} & \frac{t_1^2}{3}(t_1 - t_0)^2 \end{pmatrix}. \end{aligned}$$

The first trial is to check if the Gramian has full rank, by taking its determinant:

$$\det W(t_0, t_1) = (t_1 - t_0)^4 \cdot \frac{t_1^2}{12}. \tag{2.2}$$

Since $t_1 > t_0 > 0$, the determinant is nonzero, and therefore, the Gramian has full rank. So, the entire space $\mathbb{R}^2$ is reachable.

Another way to determine the image of the Gramian is to use elementary column/row operations to obtain a matrix of a triangular (stair-case) form:

$$W(t_0, t_1) \underbrace{\begin{pmatrix} 1 & -t_1 \frac{(t_1 - t_0)}{2} \\ 0 & 1 \end{pmatrix}}_{=:P} = \begin{pmatrix} 1 & 0 \\ t_1 \frac{(t_1 - t_0)}{2} & \frac{t_1^2}{12}(t_1 - t_0)^2 \end{pmatrix}.$$

Since P is nonsingular, the rank of W and that of the matrix in the right-hand side are the same.

Exercise 2.3 (LTI case). Under what conditions on b the system:

$$\dot{x}(t) = Ax(t) + bu(t) = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} x(t) + bu(t), \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

is completely reachable?

[1] Note that $W(t_0, t_1)$ has to be symmetrical and positive semidefinite for $t_0 < t_1$

Note that in this case the system is LTI, and therefore,

$$\text{Im}W(t_0, t_1) = \text{Im}\ [B, AB, ..., A^{n-1}B] = \text{Im}\ \Gamma$$

We want the system to be completely reachable, that is we want that Im $\Gamma = \mathbb{R}^3$. The most straightforward method to solve this exercise is to calculate Γ and check when det $\Gamma \neq 0$:

$$\Gamma = \begin{bmatrix} b_1 & \lambda b_1 + b_2 & \lambda^2 b_1 + 2\lambda b_2 + b_3 \\ b_2 & \lambda b_2 + b_3 & \lambda^2 b_2 + 2\lambda b_3 \\ b_3 & \lambda b_3 & \lambda^2 b_3 \end{bmatrix} \Rightarrow \quad \det \Gamma = -b_3^3$$

Therefore, the system is completely reachable if and only if $b_3 \neq 0$.[2]

Exercise (LTI case with multiple inputs). Consider the LTI system:

$$\dot{x}(t) = \underbrace{\begin{pmatrix} -2 & 2 & 0 \\ 0 & -0.5 & 0.5 \\ 1 & -1.5 & -0.5 \end{pmatrix}}_{A} x(t) + \underbrace{\begin{pmatrix} 0 & 2 \\ 0.5 & 0.5 \\ 0.5 & -1.5 \end{pmatrix}}_{B} u(t) \tag{2.3}$$

Obtain the reachable subspace for this system.

A procedure to obtain the reachable subspace $\mathcal{R}$ is as follows.

(1) Form the reachability matrix

$$\Gamma := [B, AB, A^2B] = \begin{bmatrix} 0 & 2 & 1 & -3 & -2 & 4 \\ 0.5 & 0.5 & 0 & -1 & -0.5 & 1.5 \\ 0.5 & -1.5 & -1 & 2 & 1.5 & -2.5 \end{bmatrix}. \tag{2.4}$$

The reachable subspace $\mathcal{R}$ is ImΓ, but in this example, it is not so trivial to figure out what ImΓ is.

(2) By elementary column/row operations, transform Γ into a triangular form, for which we can easily see the image space. To this end:

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[2] It is also possible to solve this exercise using a small trick. Note that the matrix A is in Jordan canonical form with a single eigenvalue λ with multiplicity equal to 3. Let us introduce the auxiliary variable $\tilde{x}(t)$ as:

$$\tilde{x}(t) = e^{-\lambda t} x(t)$$

Since $e^{-\lambda t} \neq 0\ \forall t$, the variables $x(t)$ and $\tilde{x}(t)$ have the same reachability properties. We have that:

$$\dot{\tilde{x}}(t) = -\lambda e^{-\lambda t} x(t) + e^{-\lambda t}\dot{x}(t) = S\tilde{x}(t) + bu(t)e^{-\lambda t} = Sx + b\tilde{u}(t)$$

where the matrix S is defined as:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So, we can study the reachability of $\tilde{x}$ which is easier to compute:

$$\tilde{\Gamma} = [b, Sb, S^2 b] = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & 0 \\ b_3 & 0 & 0 \end{bmatrix}$$

and clearly $\tilde{\Gamma}$ is full rank if and only if $b_3 \neq 0$.

(a) Swap the first and the second rows. This corresponds to a matrix multiplication from the left of Γ:

$$\Gamma_1 := \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_1} \Gamma = \begin{bmatrix} 0.5 & 0.5 & 0 & -1 & -0.5 & 1.5 \\ 0 & 2 & 1 & -3 & -2 & 4 \\ 0.5 & -1.5 & -1 & 2 & 1.5 & -2.5 \end{bmatrix}. \tag{2.5}$$

(b) Make the (1,1) entry become 1. This corresponds to a matrix multiplication from the left of Γ_1:

$$\Gamma_2 := \underbrace{\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{P_2} \Gamma_1 = \begin{bmatrix} 1 & 1 & 0 & -2 & -1 & 3 \\ 0 & 2 & 1 & -3 & -2 & 4 \\ 0.5 & -1.5 & -1 & 2 & 1.5 & -2.5 \end{bmatrix}. \tag{2.6}$$

(c) In order to make the (3,1) entry become zero, we use

$$\text{(new third row)} = \text{(first row)} - 2 \times \text{(third row)}.$$

This corresponds to a matrix multiplication from the left of Γ_3:

$$\Gamma_3 := \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}}_{P_3} \Gamma_2 = \begin{bmatrix} 1 & 1 & 0 & -2 & -1 & 3 \\ 0 & 2 & 1 & -3 & -2 & 4 \\ 0 & 4 & 2 & -6 & -4 & 8 \end{bmatrix}. \tag{2.7}$$

(d) We can easily see that

$$\text{(third row)} = 2 \times \text{(second row)}.$$

So, in order to make the third row to be zero, we multiply Γ_3 by a matrix from the left:

$$\Gamma_4 := \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}}_{P_4} \Gamma_3 = \begin{bmatrix} 1 & 1 & 0 & -2 & -1 & 3 \\ 0 & 2 & 1 & -3 & -2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.8}$$

Now it is trivial that $\text{Im}\Gamma_4 = \text{span}\{e_1, e_2\}$.

(3) Obtain the reachable subspace by tracing back the matrix multiplications:

$$\begin{aligned} \text{Im}\Gamma &= \text{Im}(P_1^{-1}P_2^{-1}P_3^{-1}P_4^{-1}\Gamma_4) \\ &= \text{Im}((P_4P_3P_2P_1)^{-1}\Gamma_4) \\ &= \text{Im}((P_4P_3P_2P_1)^{-1}[e_1, e_2]). \end{aligned}$$

By a calculation,

$$(P_4P_3P_2P_1)^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0 \\ 0.5 & -1 & 0.5 \end{bmatrix}. \tag{2.9}$$

Therefore,

$$\text{Im}\Gamma = \text{span}\left\{ \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}. \tag{2.10}$$

Exercise (Decomposition theorem). Let us consider the following LTI system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} u(t) = Ax(t) + Bu(t)$$

The reachability matrix is:

$$\Gamma = [B, AB, A^2B, A^3B] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

It is easy to see that Γ has rank 3, and so the system is not completely reachable. Therefore we can write:

$$\mathbb{R}^4 = \mathcal{R} \oplus \mathcal{V}$$

where $\mathcal{R}$ is the reachable subspace, and $\mathcal{V}$ is a complement. Here we want to give an example of the decomposition theorem described in the Example 3.2.9 in the compendium. To this end, we want to express the system in a new basis such that the first 3 vectors span $\mathcal{R}$, and the remaining one spans $\mathcal{V}$. Let us determine a basis of the reachable space $\mathcal{R}$ and for a complement $\mathcal{V}$ in $\mathbb{R}^4$:

$$\mathcal{R} = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad \mathcal{V} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Then, we define the transformation matrix T in the following way:

$$T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

In the new basis (or by setting $x := Tz$) the system matrices have the following form:

$$\begin{aligned} \bar{A} &= T^{-1}AT = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}, \\ \bar{B} &= T^{-1}B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}. \end{aligned}$$

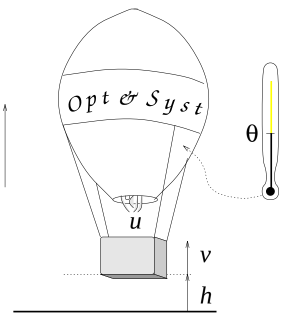
It is easy to see that the subsystem $(\bar{A}_{11}, \bar{B}_1)$ is completely reachable, and its reachable space is $\mathcal{R}$. The subsystem $(\bar{A}_{22}, 0)$ is the unreachable part of the system. Its dynamics cannot be influenced by the input $u(t)$, and depend only on the initial state x_0.

3. Observability examples

Exercise 2.5.

A rough description of the movement of a hot air balloon is

$$\begin{cases} \dot{\theta} &= -\frac{1}{\alpha}\theta + u \\ \dot{v} &= -\frac{1}{\beta}v + \sigma\theta + \frac{1}{\beta}w \\ \dot{h} &= v, \end{cases} \tag{3.1}$$



where

θ	: temperature
u	: heating
v	: vertical velocity
h	: height
w	: vertical wind velocity
α, β, σ	: given positive constants

(a) Assume that the wind velocity w is constant but unknown. Is it then possible to reconstruct θ and w through observations of the height h?

To answer this question, we set up a problem on observability of θ and w. This means that the state vector should include θ and w, and the output is h.

$$\begin{bmatrix} \dot{\theta} \\ \dot{v} \\ \dot{h} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\alpha} & 0 & 0 & 0 \\ \sigma & -\frac{1}{\beta} & 0 & \frac{1}{\beta} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ v \\ h \\ w \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \tag{3.2}$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ v \\ h \\ w \end{bmatrix}. \tag{3.3}$$

To be able to reconstruct θ and w from h, the condition is

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \perp \ker\Omega, \tag{3.4}$$

where Ω is the observability matrix:

$$\Omega := \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \sigma & -\frac{1}{\beta} & 0 & \frac{1}{\beta} \\ -\left(\frac{\sigma}{\alpha} + \frac{\sigma}{\beta}\right) & \frac{1}{\beta^2} & 0 & -\frac{1}{\beta^2} \end{bmatrix}. \tag{3.5}$$

Since Ω is full rank, $\ker\Omega = \{0\}$, and therefore, we can reconstruct θ and w from h.

(b) Assume that $w = 0$. Is the system completely reachable?

In (3.2), we delete the column/row related to w:

$$\begin{bmatrix} \dot{\theta} \\ \dot{v} \\ \dot{h} \end{bmatrix} = \begin{bmatrix} -\frac{1}{\alpha} & 0 & 0 \\ \sigma & -\frac{1}{\beta} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ v \\ h \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u. \tag{3.6}$$

The reachability matrix is

$$\Gamma := [B, AB, A^2B] = \begin{bmatrix} 1 & -\frac{1}{\alpha} & \frac{1}{\alpha^2} \\ 0 & \sigma & -\left(\frac{\sigma}{\alpha} + \frac{\sigma}{\beta}\right) \\ 0 & 0 & \sigma \end{bmatrix}, \tag{3.7}$$

which is (obviously!) full rank. Hence the system is completely reachable.

Exercise 2.7. Consider the system:

$$\begin{cases} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \\ y(t) &= C(t)x(t) + D(t)u(t) \end{cases}$$

Denote with $\Phi(t, s)$ the transition matrix, and define the observability Gramian:

$$M(t_0, t_1) \triangleq \int_{t_0}^{t_1} \Phi^T(t, t_0) C^T(t) C(t) \Phi(t, t_0) dt$$

(a): Show that we can distinguish between the initial points $x(t_0) = a$ and $x(t_0) = b$ iff $b - a \notin \ker M(t_0, t_1)$.

Define the linear operator $T : \mathbb{R}^n \to \mathcal{Y}$, where $\mathcal{Y}$ is the space of m-dimensional, square-integrable functions on $[t_0, t_1]$, as:

$$(Tx_0)(t) \triangleq C(t)\Phi(t, t_0)x_0$$

The adjoint operator, $T^* : \mathcal{Y} \to \mathbb{R}^n$ is then defined as (see compendium):

$$T^*y = \int_{t_0}^{t_1} \Phi^T(t, t_0) C^T(t) y(t) dt$$

From the compendium we know that it is not possible to distinguish between the two initial conditions if and only if

$$T(a) = T(b) \Rightarrow \quad T(a - b) = 0 \Rightarrow \quad a - b \in \ker T$$

Since $\mathcal{Y}$ is an infinite dimensional space, the operator T does not have a finite dimensional matrix representation. So, instead of considering T, let us consider the linear operator $T^*T : \mathbb{R}^n \to \mathbb{R}^n$. T^*T is defined as:

$$T^*T(x_0) = \underbrace{\int_{t_0}^{t_1} \Phi^T(t, t_0) C^T(t) C(t) \Phi(t, t_0) dt}_{M(t_0, t_1)} \, x_0.$$

Therefore, it is enough to show that $\ker T = \ker T^*T$. Let us first show that $\ker T \subseteq \ker T^*T$. Suppose that $Tx = 0$, then we have that:

$$T^*Tx = T^*0 = 0 \quad \Rightarrow \ker T \subseteq \ker T^*T$$

Next, let us show that ker $T^*T \subseteq$ ker T. Suppose that $T^*Tx = 0$. Then we have that:

$$(x, T^*Tx)_{\mathbb{R}^n} = 0 \Rightarrow (Tx, Tx)_{\mathcal{Y}} = 0 \Rightarrow Tx = 0 \quad \Rightarrow \ker T^*T \subseteq \ker T$$

Hence, ker $T^*T =$ ker T

(b): Suppose that the matrices A and C are constant. Consider the observability matrix Ω and show that ker $M(t_0, t_1) =$ ker Ω.

Fix t_0 and t_1 and let $M \triangleq M(t_0, t_1)$. Proceeding as before, let us first show that ker $M \subseteq$ ker Ω. Let $a \in$ ker M. Then:

$$0 = a^T M a = \int_{t_0}^{t_1} a^T e^{A^T(t-t_0)} C^T C e^{A(t-t_0)} a dt = \int_{t_0}^{t_1} \|Ce^{A(t-t_0)}a\|^2 dt$$

Since the integrand is non-negative and continuous it must be that:

$$Ce^{A(t-t_0)}a \equiv 0 \quad \forall t \in [t_0, t_1]$$

If we take the Taylor expansion of the exponential we get:

$$\sum_{k=0}^{\infty} \frac{1}{k!}(t-t_0)^k CA^k a \equiv 0 \Rightarrow CA^k a = 0 \quad k = 1, ..., n \Rightarrow a \in \ker\Omega$$

Next, let us show that ker $\Omega \subseteq$ ker M. This is equivalent to show that Im $M \subseteq$ Im Ω^T. (Note that $M = M^T$.) Suppose then that $a \in$ Im M. Then, there is a vector $x \in \mathbb{R}^n$ such that:

$$a = Mx = \sum_{k=0}^{\infty} (A^T)^k C^T \int_{t_0}^{t_1} \frac{1}{k!}(t-t_0)^k C e^{A(t-t_0)} x dt$$

Therefore we have that $a \in$ Im $[C^T, A^T C^T, (A^T)^2 C^T, \cdots]$. But for the Cayley-Hamilton theorem we get that:

$$a \in \text{Im}[C^T, A^T C^T, \cdots, (A^T)^{n-1} C^T] = \text{Im}\Omega^T$$

So, in conclusion we have that ker $M =$ ker Ω

(c): Define the "quiet" subspace S as $S \triangleq$ ker Ω. Partition the state space as $R^n = V \oplus S$ where $V = S^\perp$. Then, show that S is A-invariant and determine the block-matrix representation of the system corresponding to the subspace above. Apply the previous result to the following system:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 2 & 4 & 3 \\ 4 & 2 & 3 \\ 4 & 4 & 4 \end{pmatrix} x(t) + \begin{pmatrix} 1 & 0 \\ -1 & 3 \\ 0 & 2 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 2 & 2 & -3 \end{pmatrix} x(t) \end{aligned}$$

Let us consider the quiet, or unobservable subspace S. We have that:

(1) S **is** A**-invariant**. Assuming $x \in \ker\Omega$, we will prove $Ax \in \ker\Omega$.

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x = 0 \implies \Omega A x = \begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^n \end{bmatrix} x = 0.$$

The last equality is a consequence of the Cayley-Hamilton theorem.

(2) **Block-matrix representation of the system**. Suppose that S has dimension d. Then, choose a basis for $\mathbb{R}^n$ such that the first $n-d$ vector are a basis for V, and the remaining d vectors are a basis for S. Then, we can partition each vector x as $x = [x_1', x_2']'$ with x_2 d-dimensional. We will show that in this basis the system has the structure:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} x(t) + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u(t), \\ y(t) &= \begin{pmatrix} C_1 & 0 \end{pmatrix} x(t) + Du(t). \end{aligned}$$

An arbitrary vector $x \in S$ can be written as:

$$x = \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$

and if we multiply it by A:

$$Ax = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{12}x_2 \\ A_{22}x_2 \end{bmatrix} \quad \Rightarrow A_{12} = 0 \quad \text{since } Ax \in S$$

Besides, we have that:

$$Cx = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = 0 \quad \text{since } x \in S \quad \Rightarrow C_2 = 0$$

(3) **An example**: Compute the observability matrix Ω:

$$\Omega = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, let us find a basis for the unobservable subspace

$$\ker \Omega = \{x : 2x_1 + 2x_2 - 3x_3 = 0\} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

A basis vector for the subspace V can be computed with the cross product:

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} = \begin{vmatrix} e_x & e_y & e_z \\ 1 & -1 & 0 \\ 0 & 3 & 2 \end{vmatrix} = \begin{bmatrix} -2 \\ -2 \\ 3 \end{bmatrix}.$$

We can then build the matrix T for the change of basis:

$$T = \begin{bmatrix} -2 & 1 & 0 \\ -2 & -1 & 3 \\ 3 & 0 & 2 \end{bmatrix}$$

Let us call $z = T^{-1}x$ the new state variable. We have that:

$$\begin{aligned} \dot{z}(t) &= \underbrace{\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline -3 & -2 & 18 \\ -2 & 0 & 10 \end{array}\right)}_{T^{-1}AT} z(t) + \underbrace{\left(\begin{array}{cc} 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right)}_{T^{-1}B} u(t), \\ y(t) &= \underbrace{\left(\begin{array}{c|cc} -17 & 0 & 0 \end{array}\right)}_{CT} z(t). \end{aligned}$$

(d): Show that the pair (A, C) is completely observable if and only if[3]

$$Cq \neq 0 \quad \text{for any right eigenvector } q \text{ of } A \text{ (i.e., } Aq = \lambda q). \tag{3.8}$$

First, assume that there is some right eigenvector q of A satisfying $Cq = 0$, and prove that (A, C) is not completely observable. In this case,

$$\Omega q = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} q = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This means that Ω is column rank deficient, and (A, C) is not completely observable.

Next, conversely, assume that the system is not completely observable. Then, the system can be transformed, by a variable change $x = Tz$, into the one having the following block structure:

$$\dot{z}(t) = \underbrace{\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}}_{\tilde{A} := T^{-1}AT} z(t), \quad y(t) = \underbrace{\begin{pmatrix} C_1 & 0 \end{pmatrix}}_{\tilde{C} := CT} z(t).$$

Defining

$$\tilde{q} := \begin{bmatrix} 0 \\ w \end{bmatrix}, \tag{3.9}$$

where w is any eigenvector of A_{22}, we can verify

$$\begin{aligned} \tilde{A}\tilde{q} = \lambda \tilde{q} &\implies A(T\tilde{q}) = \lambda(T\tilde{q}) \\ \tilde{C}\tilde{q} = 0 &\implies C(T\tilde{q}) = 0 \end{aligned}$$

Thus, there exists a right eigenvector $q := T\tilde{q}$ of A that satisfies $Cq = 0$.

[3]The condition is known to be the PBH (Popov-Belewitch-Hautus) test. The PBH test for reachability is as follows: The pair (A, B) is completely reachable if and only if $q^*B \neq 0$ for any left eigenvector q^* of A (i.e., $q^*A = \lambda q^*$).