SYSTEMTEORI - KALMAN FILTER VS LQ CONTROL

1. Optimal regulator with noisy measurement

Consider the following system:

$$\dot{x} = Ax + Bu + w, \quad x(0) = x_0$$

where $w(t)$ is white noise with $Ew(t) = 0$, and x_0 is a stochastic variable with $Ex_0x_0^T = P_0$. Consider the following cost function:

$$J(u) = E\left\{\int_{t_0}^{t_1}[x^TQx + u^TRu]dt + x^T(t_1)Sx(t_1)\right\}$$

where $R > 0$, $Q \geq 0$ and $S \geq 0$. The problem of determining for each t the input $u(t)$ as a function of the past such that the cost function is minimized is called the stochastic state feedback regulator problem. Note that since all the variables are stochastic, we consider the average of the usual cost function.

It is possible to prove that the solution of the stochastic state feedback regulator problem is the same as in the deterministic case. The presence of white noise does not alter the solution, except to increase the minimal value of the cost function. That is, the optimal control input, $u(t)$, is given by:

$$u(t) = -R^{-1}B^TP(t)x(t) \tag{1.1}$$

where $P(t)$ is the solution of the (RE):

$$\dot{P} = -A^TP - PA + PBR^{-1}B^TP - Q \tag{1.2}$$

$$P(t_1) = S \tag{1.3}$$

Until now we have considered the unrealistic situation that we can somehow measure the state vector $x(t)$ in order to compute the optima control input $u(t)$. A more realistic situation is the case of output feedback, where we use the measured variable $y(t)$ to make an estimation $\hat{x}(t)$ of the state, see section 6.2 in the compendium. Now, we want to formulate the optimal linear regulator problem when the observation of the system are noisy. That is, consider the system:

$$\dot{x}(t) = Ax(t) + Bu(t) + w_1(t), \quad x(0) = x_0$$

where x_0 is a stochastic vector with zero mean and covariance P_0. The observed variable is given by:

$$y(t) = Cx(t) + Dw_2(t)$$

where $Ew_2 = 0$ and $Ew(t)w^T(s) = \delta(t-s)I$ Then, the stochastic optimal output feedback regulator problem is the problem of finding the functional $u(t) = f[y(\tau), t_0 \leq \tau \leq t]$ such that the cost function:

$$J(u) = E\left\{\int_{t_0}^{t_1}[x^TQx + u^TRu]dt + x^T(t_1)Sx(t_1)\right\}$$

is minimized.

It is possible to prove that **the solution of the stochastic optimal output feedback regulator problem is the same as the solution of the corresponding optimal state feedback regulator problem, eq. (1.1) and (1.2), except that in the control law (1.1) the state $x(t)$ is replaced with the Kalman filter estimator $\hat{x}(t)$**, that is the optimal control input is chosen as:

$$u(t) = -R^{-1}B^T P(t)\hat{x}(t) \tag{1.4}$$

where $P(t)$ is the solution of the (RE):

$$\dot{P} = -A^T P - PA + PBR^{-1}B^T P - Q \tag{1.5}$$
$$P(t_1) = S \tag{1.6}$$

The estimate $\hat{x}(t)$ is obtained as the solution of

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(t)[y(t) - C\hat{x}(t)] \tag{1.7}$$
$$\hat{x}(t_0) = 0 \tag{1.8}$$

where

$$L(t) = P(t)C^T R^{-1} \tag{1.9}$$

and $P(t)$ is the solution of the (RE):

$$\dot{P} = AP + PA^T - PC^T R^{-1} CP + Q \tag{1.10}$$
$$P(t_0) = P_0 \tag{1.11}$$

2. Examples

2.1. **Es. 1 (from Tentamen 20 oktober 1998).** Consider the following differential equation describing a simple electrical circuit:

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dw}{dt}$$

where i is the current and w is a standard wiener process. Suppose we simplify it, by setting $R = 0$. By substituting the appropriate numerical values we get:

$$\frac{d^2 i}{dt^2} + i = \sqrt{3}v$$

where v is a normal distributed white noise with

$$Ev(t) = 0, \quad Ev(t)v(s) = \delta(t-s)$$

The measurement, $s(t)$, are taken with an ampere meter where

$$s(t) = i(t) + e(t)$$

where $e(t)$ is a normal distributed white noise, uncorrelated from the system noise $v(t)$, and such that:

$$Ee(t) = 0, \quad Ee(t)e(s) = \delta(t-s)$$

Your assignment is to determine a steady-state Kalman filter to estimate the current, $i(t)$, from the noisy observation, $\{s(\tau), 0 \le \tau \le t\}$, by using the associated linear system.

First, let us rewrite the problem in the standard form. Let $x_1 = i$, and $x_2 = di/dt$. The, we have that:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \sqrt{3}v\end{aligned}$$

that is:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix} v = Ax + Bv$$

and

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + e(t) = Cx(t) + De(t)$$

In section 9.2.3 of the compendium it is proven that the covariance matrix of the estimation error, $P(t)$, tends to a limit P as $t \to \infty$. If (A, B) is completely reachable, and (A, C) is completely observable, then P is the unique positive definite symmetric solution of the (ARE):

$$AP + PA' - PC'(DD')^{-1}CP + BB' = 0$$

and consequently the Kalman filter gain tends to

$$K = PC'(DD')^{-1}$$

In our case the system is a minimal realization. In fact, we have that the reachability and observability matrices are:

$$\Gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and they both are full rank.

In order to solve the (ARE) we define the matrix P as

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$$

and we plug it in the (ARE):

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} = 0$$

So, we get the following system:

$$\begin{cases} 2p_2 - p_1^2 &= 0 \\ p_3 - p_1 - p_1p_2 &= 0 \\ -2p_2 - p_2^2 + 3 &= 0 \end{cases}$$

By solving the system, and taking the solution that gives a positive definite P we get:

$$P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & 2\sqrt{2} \end{bmatrix}$$

Then, the steady state Kalman gain is:

$$K = PC'R^{-1} = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$

Note that the estimator system matrix is:

$$A - KC = \begin{bmatrix} -\sqrt{2} & 1 \\ -2 & 0 \end{bmatrix}$$

which has the following eigenvalues:

$$\chi_{A-KC}(\lambda) = (\lambda + \frac{\sqrt(2)}{2} + 2 - \frac{1}{2}) \Rightarrow \lambda_{1,2} = -\frac{1}{\sqrt{2}} \pm i\frac{\sqrt{3}}{\sqrt{2}}$$

that is, $A - KC$ is a stability matrix.

2.2. **Es. 2 (from Tentamen 1981).** Consider the system:

$$\begin{aligned} \dot{x} &= x + 2u \\ y &= x + e \end{aligned}$$

where $x(0) \in \mathcal{N}(0, \sqrt{3})$, and e is a process of independent, normal distributed increments, with $Ee = 0$, $Ee^2 = 2$. We want to determine a control input $\hat{u}(t, \hat{x})$ such that the cost function:

$$J(u) = E(\int_t^T 2x^2 + u^2 dt)$$

is minimized.

a: Determine the optimal feedback gain, $K(t)$, and the optimal estimator gain, $L(t)$.

b: Suppose there is an additive, non-white noise with independent, normal distributed increments and with correlation function $e^{-|\tau|}$. That is, the process is described by:

$$\dot{x} = x + 2u + w, \quad x(0) \in \mathcal{N}(0, \sqrt{3})$$

Let $T = \infty$, and determine the optimal steady-state gains K and L under the assumption that e and w are independent.
Hint: Use the following equation for $w(t)$:

$$dw(t) = -wdt + dv$$

where $E(v) = 0$ and $E(v)^2 = 2$

c: Is the overall system stable?

a: By using the separation principle, we can first design an optimal feedback gain, where we neglect the measurement noise. Then, we can design an optimal estimator, where instead we take into account the noise.

The problem in the usual LQ framework becomes:

$$\min_u J(u) = \int_t^T (2x^2 + u^2)dt$$

subject to

$$\dot{x} = x + 2u$$

The optimal control input is $\hat{u} = Kx$ where:

$$K = -R^{-1}B^T P = -2P$$

and P is the solution of the (RE)

$$\begin{aligned} \dot{P} &= -A^T P - PA + PBR^{-1}B^T P - Q \\ P(T) &= S \end{aligned}$$

that is

$$\begin{aligned} \dot{P} &= -2P + 4P^2 - 2 \\ P(T) &= 0 \end{aligned}$$

In order to solve the previous nonlinear scalar differential equation, we suppose that $P(t)$ has the following form:

$$P(t) = k\frac{\dot{z}(t)}{z(t)}$$

If we substitute it in the original differential equation we get that:

$$k\frac{\ddot{z}}{z} - k\frac{\dot{z}^2}{z^2} = -2k - \frac{\dot{z}}{z} + 4k^2\frac{\dot{z}^2}{z}^2 - 2$$

$$\Rightarrow k\frac{\ddot{z}}{z} + \frac{\dot{z}^2}{z^2}(-k - 4k^2) + 2k\frac{\dot{z}}{z} + 2$$

If we take $k = -1/4$ the expression simplifies and finally we get that:

$$\ddot{z} + 2\dot{z} - 8z = 0 \Rightarrow z(t) = c_1e^{-4t} + c_2e^{2t}$$

$$\Rightarrow P(t) = -\frac{1}{4}\frac{-4c_1e^{-t} + 2c_2e^{2t}}{c_1e^{-4t} + c_2e^{2t}} = \frac{1}{2}\frac{2e^{-6t} - c}{e^{-6t} + c}$$

By imposing $P(T) = 0$, we finally get:

$$P(t) = \frac{e^{-6(t-T)} - 1}{e^{-6(t-T)} + 2} =$$

and hence, the optimal control input is:

$$u(t, x) = -2\frac{e^{-6(t-T)} - 1}{e^{-6(t-T)} + 2}x$$

Next, we have to find the optimal estimator (Kalman-Buchy filter). In this case we do not consider the input u. We have that:

$$\begin{aligned} \dot{x} &= x \\ y &= x + \sqrt{2}v \end{aligned}$$

where $x(0) \in \mathcal{N}(0, \sqrt{3})$, and $v(t)$ is white noise uncorrelated from $x(0)$ and with $Ev(t)v(s) = \delta(t - s)$. This is a standard problem with:

$$A = 1, \quad B = 0, \quad , C = 1, \quad D = \sqrt{2}$$

Therefore, the optimal estimator is given by:

$$L(t) = P(t)C^TR^{-1}$$

with $P(t)$ solution to the (RE):

$$\begin{aligned} \dot{P} &= A^TP + PA - PC^TR^{-1}CP + Q \\ P(0) &= P_0 \end{aligned}$$

where

$$R = DD^T, \qquad Q = BB^T$$

By inserting the numerical values, we get:

$$L(t) = 2P(t), \quad R = 2, \quad , Q = 0, \quad , P_0 = 3$$

and the (RE) becomes:

$$\begin{aligned} \dot{P} &= 2P - P^2/2 \\ P(0) &= 3 \end{aligned}$$

We can solve the differential equation by using the following substitution $P(t) = \frac{1}{z(t)}$:

$$\begin{aligned} \Rightarrow \dot{z} + 2z - 1/2 &= 0 \\ z(0) &= 1/3 \end{aligned}$$

which can be easily solved:

$$z(t) = \frac{1}{12}e^{-2t} + \frac{1}{4}$$

And so we have that the optimal gain is:

$$L(t) = \frac{6}{e^{-2t} + 3}$$

and the optimal estimator is:

$$\begin{aligned} \frac{d\hat{x}}{dt} &= \hat{x} + 2u + L(t)(y - \hat{x}) \\ \hat{x}(0) &= 0 \end{aligned}$$

b: We have that the overall system can be described by the following equations:

$$\begin{aligned} \dot{x} &= x + w + 2u \\ \dot{w} &= -w + v \\ y &= x + e \end{aligned}$$

with x_0 and w_0 independent processes such that:

$$Ex_0 = Ew_0 = 0, \quad E\begin{bmatrix} x_0 \\ w_0 \end{bmatrix}\begin{bmatrix} x_0 & w_0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

We introduce the new variable $z = [x^T, w^T]^T$ and we determine the gains K and L by using the separation principle, like before. First, we consider the previous system without measurement noise, $e(t)$, and we solve the corresponding LQ problem. That is, we have the following linear system:

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} z + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} z \end{aligned}$$

and we want to minimize the cost function:

$$J(u) = \int_0^\infty (2y^2 + u^2)dt$$

To solve this LQ problem we have to check if the realization is minimal. The reachability and observability matrices are:

$$\Gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

Unfortunately, the system is not completely reachable. Clearly, we can not control the noise $w(t)$! However, since the system is asymptotically stable, the (ARE) has anyway a positive definite solution. So, the optimal solution, $\hat{u}$, is given by:

$$\hat{u} = -Kz \quad \text{with } K = B^T P$$

and where the matrix P is the unique positive solution of the (ARE):

$$A'P + PA - PBB'P + Q = 0$$

where

$$Q = \begin{bmatrix} 2 & 0 \\ 00 & \end{bmatrix}$$

By inserting the numerical values, we get:

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

So, we get the following system:

$$\begin{cases} 2p_1^2 + 2 - 4p_1^2 & = & 0 \\ p_2 + p_1 - p_2 - 4p_1p_2 & = & 0 \\ 2(p_2 - p_3) - 4p_2^2 & = & 0 \end{cases}$$

By solving the system, and taking the solution that gives a positive definite P we get:

$$P = \begin{bmatrix} 1 & 1/4 \\ 1/4 & 1/8 \end{bmatrix} > 0$$

and so, the optimal gain, K, is:

$$K = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/4 \\ 1/4 & 1/8 \end{bmatrix} = \begin{bmatrix} 2 & 1/2 \end{bmatrix}$$

Next, we have to determine the optimal observer, that is the Kalman-Bucy filter. We have the following system:

$$\begin{aligned} \dot{z} &= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} z + \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} v \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} z + \sqrt{2}\eta \end{aligned}$$

Therefore, we have a standard problem with the following numerical data:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = \sqrt{2}$$

$$R = DD' = 2, \quad Q = BB' = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Note that in this case the realization is minimal. In fact, we have that the reachability and observability matrices are:

$$\Gamma = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} \end{bmatrix}, \quad \Omega = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

So, we know we have an unique positive solution to the (ARE). The steady-state Kalman filter gain L is given by:

$$L = PC^T R^{-1}$$

and where the matrix P is the unique positive solution of the (ARE):

$$AP + PA' - PC'R^{-1}CP + Q = 0$$

By inserting the numerical values, we get:

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = 0$$

with solution (we obtained it numerically)

$$P = \begin{bmatrix} 4.3947 & 0.4337 \\ 0.4337 & 0.9530 \end{bmatrix}$$

and the Kalman gain becomes

$$K = PC^T R^{-1} = \begin{bmatrix} 2.1974 & 0.2168 \end{bmatrix}$$

c: The overall system is stable according to the theory studied in chapter 8 and 9 of the compendium. However, we can check it by computing the eigenvalues of the matrix:

$$\begin{bmatrix} A + BK & -BK \\ 0 & A - LC \end{bmatrix}$$

That is, the eigenvalues of $A + BK$ and of $A - LC$:

$$\chi_{A+BK}(\lambda) = (\lambda + 3)(\lambda + 1) \Rightarrow \lambda_{1,2} = -3, -1$$

$$\chi_{A-LC}(\lambda) = (\lambda + 1)^2 + 1 \Rightarrow \lambda_{1,2} = -1, -1.1974 - 1$$

and so the overall system is asymptotically stable.