KTH Matematik

Exam May 23 2006 in 5B1742 Mathematical Systems Theory.

Examiner: Ulf Jönsson, tel. 790 84 50.

Allowed books: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory and β mathematics handbook.

Solution methods: All conclusions should be properly motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades: 25 credits or more gives grade 3. 35 credits or more gives grade 4 and 45 or more credits gives grade 5.

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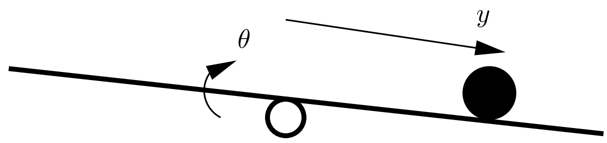
1. Consider the ball and beam system in Figure 1. The system satisfies the nonlinear



Figur 1: Ball and beam system

differential equation

$$\ddot{y} = \frac{mgr^2}{J}\sin(\theta)$$

where we assume that the physical parameters are chosen such that $mgr^2/J = 1$.

(a) Derive a linear state space representation by linearizing the above equation around the equilibrium soulution $(y, \dot{y}, \theta) = (0, 0, 0)$. Let the states be $x_1 = y$, $x_2 = \dot{y}$, and let $u = \theta$. . . . . (2p)

(b) Is the linearized system stable when $u = 0$? . . . . (1p)

(c) We are going to control the ball and beam system using a computer. For that matter we need to discretize the system using the sampling procedure in Example 2.2.1 of Lindquist and Sand (page 14). Determine the realization

$$x(t+1) = Fx(t) + Gu(t) \tag{1}$$

by using the formulas in Lindquist and Sand with $h = 1$. . . . . (1p)

(d) Is the discrete time system stable when $u = 0$? Is this surprising? . . . . (1p)

(e) Determine a state feedback that places all the closed loop poles of the discrete time system (1) in $z = 0$. . . . . (2p)

(f) Suppose we only can measure the position y at the sample time instances $t = k,\ k = 0, 1, \ldots$. Design an observer that estimates the state of (1) such that the error dynamics has all its poles in $z = 0.5$. . . . . (3p)

2. Consider the rational transfer function

$$R(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \end{bmatrix}$$

(a) Determine the standard reachable realization. .... (4p)

(b) Show that the standard reachable realization is not minimal. .... (1p)

(c) Determine a minimal realization of $R(s)$. .... (5p)

3. Consider the following T-periodic switching system

$$\dot{x} = \begin{cases} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, & t \in [kT, (k+\frac{1}{2})T) \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, & t \in [(k+\frac{1}{2})T, (k+1)T) \end{cases}$$

(a) Show that there exist states that cannot be controlled to zero over the time interval $[0, T/2)$. .... (5p)

(b) Show that any initial state can be controlled to zero over time intervals larger than $T/2$. .... (5p)

4. Solve the following optimal control problems

(a) $\min x(1)^2 + \int_0^1 u(t)^2 dt$ subj. to $\dot{x}(t) = e^{-t}u(t),\ x(0) = 1$ .... (5p)

(b) $\min \int_0^1 u(t)^2 dt$ subj. to $\dot{x}(t) = e^{-t}u(t),\ x(0) = 1,\ x(1) = 0$ .... (5p)

5. Consider a scenario where several sensors measure one and the same scalar variable d subject to additive noise

$$y(t) = d\mathbf{1} + w(t), \quad t = 0, 1, 2, 3, \ldots \tag{2}$$

where $\mathbf{1} = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}^{\mathbf{T}}$ and w is a vector of uncorrelated white noises, i.e. $Ew(t) = 0$ and $Ew(t)w(s)^T = I\delta_{t,s}$. The unknown parameter d is modeled as a stochastic variable with $Ed = 0$ and $Ed^2 = p_0$.

(a) At each time instant, we would like to compute a linear least squares estimate of d. Design a Kalman filter for this purpose. .... (3p)

(b) How does the estimate of d improve as the number of sensors increase (let N be the number of sensors and thus the length of the vectors y and $\mathbf{1}$.)
Hint: The estimator in (a) can be implemented as a system with two states. Formulate these equations such that only scalar valued computations are performed, i.e. no matrix inversion is allowed. You may use the following formulas

$$(A + UV^T)^{-1} = A^{-1} - A^{-1}U(I + V^T A^{-1} U)^{-1} V^T A^{-1}$$
$$A(I + BA)^{-1} = (I + AB)^{-1} A$$

.... (4p)

(c) In problem (a) we used all sensor information to obtain one least squares estimate of d. Suppose instead that the sensors are distributed over a large area. Then it is no longer possible to use all measurements for a central estimate that is communicated to all localtions (as in problem (a) and (b)). Instead we want to make local estimates at each sensor location.

Suppose that there is some communication between the sensors such that at location i we have avaliable the measurements

$$z_i(t) = \sum_{j \in N_i} c_{ij} y_j(t),$$

where y_j is the j^{th} component of (2) and N_i is a set of indices that determine the sensor measurements that can be used at location i. The c_{ij} are coefficients determining the weight we put on different measurements available at location j. This in particular means that $c_{ij} = 0$ if $j \notin N_i$. We assume that $i \in N_i$. Determine a recursive least squares estimator of d at node i. .... (3p)

Good luck!