KTH Matematik

Exam in SF2832 Mathematical Systems Theory
October 24, 2007
Answers and solution sketches

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1. (*a*) We have

$$\Gamma = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$$

which does not have full rank. The system is therefore not completely reachable.

(*b*) Note that

$$\begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix} = T \begin{pmatrix} -4 & 0 \\ 0 & 2 \end{pmatrix} T^{-1}, \text{ where } T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

hence

$$\begin{aligned} \Phi(t,s) &= T \begin{pmatrix} e^{-4(t-s)} & 0 \\ 0 & e^{2(t-s)} \end{pmatrix} T^{-1} \\ &= \frac{1}{2} \begin{pmatrix} e^{-4(t-s)} + e^{2(t-s)} & e^{-4(t-s)} - e^{2(t-s)} \\ e^{-4(t-s)} - e^{2(t-s)} & e^{-4(t-s)} + e^{2(t-s)} \end{pmatrix}. \end{aligned}$$

(*c*)

$$\begin{aligned} x(t) &= \Phi(t,0)x(0) + \int_0^t \Phi(t,s)Bu(s)ds \\ &= \begin{pmatrix} e^{-4t} + e^{2t} & e^{-4t} - e^{2t} \\ e^{-4t} - e^{2t} & e^{-4t} + e^{2t} \end{pmatrix} \frac{x(0)}{2} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{4}(1 - e^{-4t}) \end{aligned}$$

which goes to $\frac{1}{4}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ if $x(0) = 0$.

(*d*) From above we have that $|x(t)| \to \infty$ if $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(*e*) No, it is not possible. This follows since $x(0)$ does not belong to the reachable subspace and sinse the unreachable subspace is unstable. A direct way to see this is by considering the formula in (*c*) then

$$x(t) = \begin{bmatrix} e^{-4t} + 2e^{t} \\ e^{-4t} - 2e^{t} \end{bmatrix} \frac{1}{2} + \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \to \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}, \quad t \to \infty$$

2. (*a*) The numerical values lead to the following equations

$$\ddot{\theta}_1 = -(\theta_1 - \theta_2) - 2\theta_1 - 5u$$
$$\ddot{\theta}_2 = -(\theta_2 - \theta_1) - 2\theta_2 + 5u$$

The state space realization is therefore

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -3 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -5 \\ 5 \end{bmatrix} u = \begin{bmatrix} 0 & I \\ \bar{A} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \bar{B} \end{bmatrix} u$$

where

$$\bar{A} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix}, \qquad B = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

and I denotes the 2×2 identity matrix.

(b) The controllability matrix has the form

$$\Gamma = \begin{bmatrix} 0 & \bar{B} & 0 & \bar{A}\bar{B} \\ \bar{B} & 0 & \bar{A}\bar{B} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -5 & 0 & 20 \\ 0 & 5 & 0 & -20 \\ -5 & 0 & 20 & 0 \\ 5 & 0 & -20 & 0 \end{bmatrix}$$

which is of rank 2. The system is thus not controllable. The reachable subspace has dimension 2.

(c) We have $C = \begin{bmatrix} \tilde{C} & 0 & 0 \end{bmatrix}$ where $\bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$. We get the observability matrix

$$\Omega = \begin{bmatrix} \bar{C} & 0 \\ 0 & \bar{C} \\ \bar{C}\bar{A} & 0 \\ 0 & \bar{C}\bar{A} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

which has rank 4. The system is completely observable.

(d) According to the system dynamics governing $\theta_1 + \theta_2$ and $\theta_1 - \theta_2$ we get

$$\ddot{\theta}_1 + \ddot{\theta}_2 = -2(\theta_1 + \theta_2)$$
$$\ddot{\theta}_1 - \ddot{\theta}_2 = -4(\theta_1 + \theta_2) - 10u$$

This gives the state space model

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -10 \end{bmatrix} u := \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & \bar{A}_2 \end{bmatrix} z + \begin{bmatrix} 0 \\ \bar{B}_2 \end{bmatrix} u$$

We may now introduce the notation

$$z_{\bar{r}} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \qquad z_r = \begin{bmatrix} z_3 \\ z_4 \end{bmatrix}$$

which satisfy

$$\dot{z}_{\bar{r}} = \bar{A}_1 z_{\bar{r}}, \qquad \dot{z}_r = \bar{A}_2 z_r + \bar{B}_2 u.$$

Clearly, $z_{\bar{r}}$ is not reachable while z_r is reachable since

$$\bar{\Gamma} = \begin{bmatrix} \bar{B}_2 & \bar{A}_2\bar{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & -10 \\ -10 & 0 \end{bmatrix}$$

which has full rank.

3. (a) We use that

$$J[0] = x_0^T \int_0^\infty e^{A^T t} Q e^{At} dt x_0 = x_0^T P x_0$$

where P is the solution to the Lyapunov equation

$$A^T P + PA + Q = 0$$

We get the solution

$$P = \begin{bmatrix} 5/4 & 1/2 \\ 1/2 & 1/4 \end{bmatrix}$$

and hence $J[0] = 1/4$.

(b) We have $\min_{u \in \mathcal{U}} J[u] = x_0^T P x_0$, where P is the positive definite solution to the ARE

$$A^T P + PA + Q - PBB^T P = 0$$

Note that (A, B) is completely reachable and that $Q = C^T C$ with $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and (A, C) is completely observable. Hence the ARE has a unique positive definite solution.
We get

$$P = \begin{bmatrix} -2 + 2\sqrt{1 + \sqrt{2}} & -1 + \sqrt{2} \\ -1 + \sqrt{2} & -2 + \sqrt{2 + 2\sqrt{2}} \end{bmatrix}$$

and hence $\min_{u \in \mathcal{U}} J[u] = -2 + \sqrt{2 + 2\sqrt{2}}$.

4. (a) False. We have

$$H_2 = \begin{bmatrix} R_1 & R_2 \\ R_2 & R_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

which has rank 1. The McMillan degree is therefore 1.

(b) False. The last state is not controllable (reachable).

(c) (i) True: The linearity follows since

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(t-s)(\alpha_1 u_1(s-T) + \alpha_2 u_2(t-s-T))ds \\ &= \alpha_1 \int_{-\infty}^{\infty} h(t-s) u_1(s-T) ds + \alpha_2 \int_{-\infty}^{\infty} h(t-s) u_2(s-T) ds \\ &= \alpha_1 y_1(t) + \alpha_2 y_2(t) \end{aligned}$$

(ii) True. The time-invariance follows since if $u_\tau(t) = u(t+\tau)$ then

$$\begin{aligned} y_\tau(t) &= \int_{-\infty}^{\infty} h(t-s) u_\tau(s) ds = \int_{-\infty}^{\infty} h(t-s) u(s+\tau-T) ds \\ &= \int_{-\infty}^{\infty} h(t+\tau-s) u(s-T) ds = y(t+\tau) \end{aligned}$$

(*iii*) False. This follows since $h(t) \neq 0$, which implies that $y(t)$ depends on $u(s)$, $s \in (-\infty, \infty)$.

(*c*) True. The statement follows since

$$\Omega v = \begin{bmatrix} C \\ C(A+BKC) \\ \vdots \\ C(A+BKC)^{n-1} \end{bmatrix} v = \begin{bmatrix} Cv \\ \lambda Cv \\ \vdots \\ \lambda^{n-1} Cv \end{bmatrix} = 0$$

where λ is the eigenvalue satisfying $Av = \lambda v$.

5. (*a*) We make a series expansion using the Markov parameters

$$\begin{aligned} G(s) &= \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_1 s^{n-1} + \ldots + a_n} \\ &= \frac{1}{s^{n-m}} \frac{b_m s^n + b_{m-1}s^{n-1} + \ldots + b_0 s^{n-m}}{s^n + a_1 s^{n-1} + \ldots + a_n} \\ &= \frac{1}{s^{n-m}} \left(b_m + \frac{\bar{b}(s)}{s^n + a_1 s^{n-1} + \ldots + a_n} \right) \\ &= \frac{1}{s^{n-m}} R_{n-m} + \frac{1}{s^{n-m+1}} R_{n-m-1} + \ldots \end{aligned}$$

where $R_{n-m} = b_m$ and

$$\bar{b}(s) = (b_{m-1} - b_m a_1)s^{n-1} + \ldots + (b_0 - b_m a_m)s^{n-m} + a_{m-1}s^{n-m-1} + \ldots + a_n$$

We can conclude that

$$\begin{aligned} R_k &= CA^{k-1}B = 0, \quad k = 1, \ldots, n-m \\ R_{n-m} &= CA^{n-m-1}B \neq 0 \end{aligned}$$

This proves the result since $d = n - m$.

(*b*) The stationary state solution satisfies $\frac{d}{dt}x(t) = se^{st}\bar{x}$ and hence the state space system has the solution

$$\begin{aligned} se^{st}\bar{x} &= Ae^{st}\bar{x} + Be^{st}\bar{u} \\ \bar{y}e^{st} &= Ce^{st}\bar{x} \end{aligned}$$

If $s = z$, where z is a zero of the transfer function then $\bar{y} = 0$. This gives the equation system

$$\begin{bmatrix} -sI + A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} = 0$$

which since $\bar{u} \neq 0$ implies that the matrix has rank less than $n + 1$.

(*c*) We have $CB \neq 0$ so $d = 1$. The matrix

$$\begin{bmatrix} -2I + A & B \\ C & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -1 & -4 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

has rank 2, which proves that $z = 2$ is a zero.