KTH Matematik

Exam January 12, 2009 in SF2832 Mathematical Systems Theory.

Examiner: Xiaoming Hu, tel. 790 7180.

Allowed books: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, class notes and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your bonus) to pass the exam. The other grade limits are listed on the course homepage.

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1. Determine if each of the following statements is true or false. You must justify your answers.

(*a*) Consider $\dot{x} = Ax$ and assume $A^T = -A$, then $x = 0$ is never asymptotically stable. .... (5p)

(*b*) Given a minimal realization (A, B, C), $(A+BK, B, C)$ is also minimal with any K. .... (5p)

(*c*) Consider a single input system $\dot{x} = Ax + bu$ where $x \in R^n$. If it is controllable, then rank $A \geq n-1$. .... (5p)

(*d*) Consider a scalar transfer function

$$g(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0},$$

where $b_m \neq 0$. If $m = 0$, then the McMillan degree of $g(s)$ is n. .... (5p)

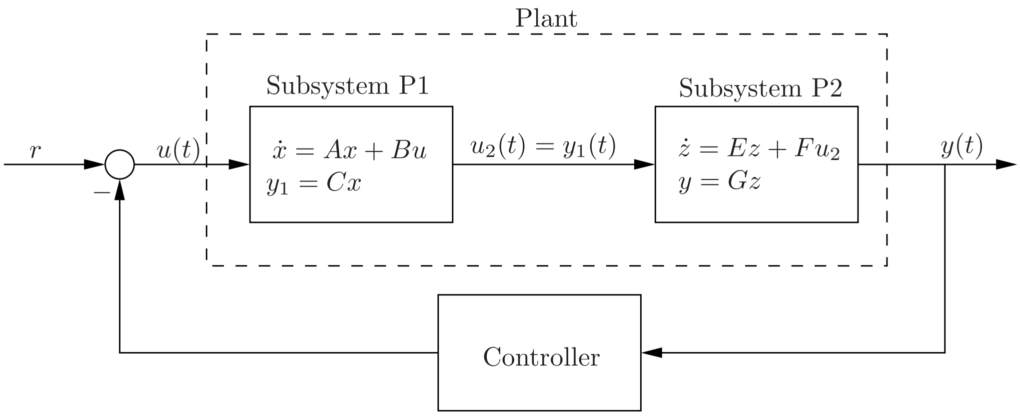
2. Consider :

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1+2a \\ -1 \end{pmatrix} u \\ y &= cx. \end{aligned}$$

(*a*) For what a is pole-assignment possible? .... (6p)

(*b*) For $a = 1$, design a feedback control $u = Fx$ such that the closed-loop poles are $\{-1, -2\}$. .... (8p)

(*c*) Find a *nonzero* output matrix c such that the resulting system is BIBO stable for all a. .... (6p)



Figur 1: A Feedback Control System

3. Consider the feedback control system shown in Figure 1, where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -3 & 2 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & 1 \end{bmatrix},$$

(a) For what α is the given state space model for Subsystem P1 minimal? ...(8p)

(b) The state space model for Plant consists of the series connection of the state space models for Subsystem P1 and Subsystem P2. Show the closed-loop plant is unstable regardless of the controller used. (Hint: transfer functions might make the proof easier). .... (12p)

4. Consider the optimal control problem

$$\min_u J = \int_0^\infty (x^T Q x + u^2) dt \quad \text{s.t.} \quad \dot{x} = Ax + Bu, \ x(0) = x_0,$$

where,

$$A = \begin{bmatrix} a_1 & 0 \\ 1 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} q^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

(a) Show for $a_2 = 0$, the associated algebraic Riccati equation (ARE) does not have a positive definite solution. .... (6p)

(b) Now suppose $q = 0$. Show that the ARE has a positive definite solution if and only if $a_1 > 0, \ a_2 > 0$. (Hint: $P > 0$ iff $P^{-1} > 0$) .... (8p)

(c) Again suppose $q = 0$. Show when the ARE has a positive definite solution, the closed-loop system has poles $\{-a_1, -a_2\}$. .... (6p)

5. At time $t = 1, 2, 3, \cdots$, an observation $y(t)$ is made of a random variable $x(t)$, with $x(0)$ being Gaussian of zero mean and constant covariance σ, namely $x(0) \sim N(0, \sigma)$. The observation error satisfies $x(t) - y(t) \sim N(0, r)$, and every time after the observation is made, $x(t)$ increases by $a \times 100\%$, where a is a constant and $-1 < a < 1$.

(a) Design a Kalman filter for the estimation of $x(t)$. ....(6p)

(b) For $a = 0$, express the covariance matrix $p(t) = E\{(x(t) - \hat{x}(t))^2\}$ in terms of σ, r. .... (7p)

(c) It is known that $p(t)$ converges to a positive semi-definite solution as t tends to ∞. Show the Kalman filter is asymptotically stable in the steady-state. ..(7p)

Good luck!