KTH Matematik

Exam January 12, 2009 in SF2832 Mathematical Systems Theory.

Examiner: Xiaoming Hu, tel. 790 7180.

Allowed books: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, class notes and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your bonus) to pass the exam. The other grade limits are listed on the course homepage.

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1. Determine if each of the following statements is true or false. You must justify your answers.

(a) Consider $\dot{x} = Ax$ and assume $A^T = -A$, then $x = 0$ is never asymptotically stable. . . . . (5p)

True since $\|x(t)\|^2 = \|x(0)\|^2$.

(b) Given a minimal realization (A, B, C), $(A+BK, B, C)$ is also minimal with any K. . . . . (5p)

False since observability can be lost by pole placement.

(c) Consider a single input system $\dot{x} = Ax + bu$ where $x \in R^n$. If it is controllable, then rank $A \geq n - 1$. . . . . (5p)

True since in the standard reachable form A must have rank at least $n - 1$.

(d) Consider a scalar transfer function

$$g(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0},$$

where $b_m \neq 0$. If $m = 0$, then the McMillan degree of $g(s)$ is n. . . . . (5p)

True since there won't be any zero/pole cancellation.

2. Consider :

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1+2a \\ -1 \end{pmatrix} u \\ y &= cx. \end{aligned}$$

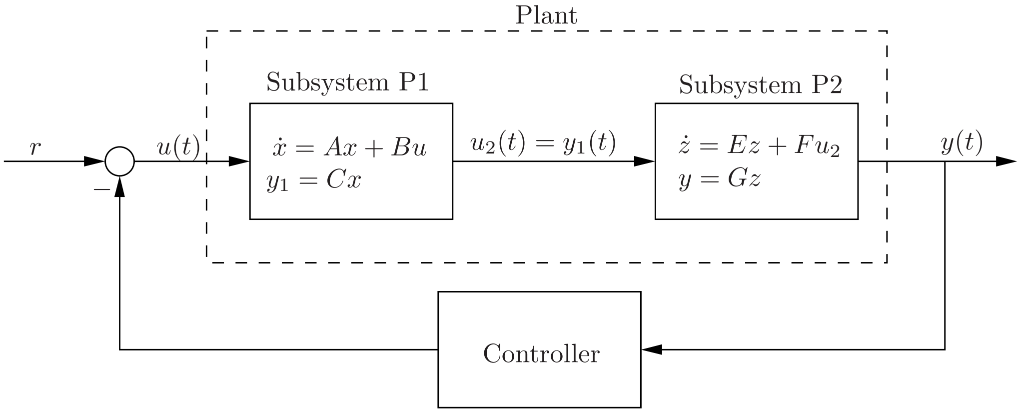
(a) For what a is pole-assignment possible? . . . . (6p)

Let $\bar{x}_1 = x_1 - x_2,\ \bar{x}_2 = x_1 + x_2$. Then one can see easily $a \neq 0,\ -1$.

(*b*) For $a = 1$, design a feedback control $u = Fx$ such that the closed-loop poles are $\{-1, -2\}$. .... (8p)
$F = -\frac{3}{2}(1\ 1)$.

(*c*) Find a *nonzero* output matrix c such that the resulting system is BIBO stable for all a.
$y = x_1 - x_2$. .... (6p)

3. Consider the feedback control system shown in Figure 1, where



Figur 1: A Feedback Control System

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -3 & 2 & 1 \end{bmatrix},$$

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 2 & 1 \end{bmatrix},$$

(*a*) For what α is the given state space model for Subsystem P1 minimal? ... (8p)
The transfer function for P1 is $\frac{(s-1)(s+3)}{s^3+\alpha}$. Thus, $a \neq -1,\ 27$.

(b) The state space model for Plant consists of the series connection of the state space models for Subsystem P1 and Subsystem P2. Show the closed-loop plant is unstable regardless of the controller used. (Hint: transfer functions might make the proof easier). .... (12p)
The transfer function for P2 is $\frac{(s+2)}{s^2-1}$. the unstable pole -1 is always canceled in the cascaded system.

4. Consider the optimal control problem

$$\min_u J = \int_0^\infty (x^T Q x + u^2) dt \quad \text{s.t.} \quad \dot{x} = Ax + Bu,\ x(0) = x_0,$$

where,

$$A = \begin{bmatrix} a_1 & 0 \\ 1 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} q^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

(a) Show for $a_2 = 0$, the associated algebraic Riccati equation (ARE) does not have a positive definite solution. . . . . (6p)

Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$. In this case one can easily show that $p_2 = p_3 = 0$.

(b) Now suppose $q = 0$. Show that the ARE has a positive definite solution if and only if $a_1 > 0,\ a_2 > 0$. . . . . (8p)

In this case one can rewrite the ARE as

$$-P^{-1}A^T - AP^{-1} = -BB^T.$$

Since $(-A, B)$ is controllable, P^{-1} is positive definite iff $-A$ is a stable matrix.

(c) Again suppose $q = 0$. Show when the ARE has a positive definite solution, the closed-loop system has poles $\{-a_1, -a_2\}$. . . . . (6p)

From the above Lyapunov equation, one can get

$$P^{-1} = \frac{1}{2}\begin{bmatrix} \frac{1}{a_1} & -\frac{1}{a_1(a_1+a_2)} \\ * & \frac{1}{a_1 a_2(a_1+a_2)} \end{bmatrix}.$$

Thus,

$$p_1 = 2(a_1 + a_2),\ p_2 = 2a_2(a_1 + a_2),$$

and $u = -B^T P x = -p_1 x_1 - p_2 x_2$. Then one can easily show that $A - BB^T P$ has eigenvalues $\{-a_1, -a_2\}$.

5. At time $t = 1, 2, 3, \cdots$, an observation $y(t)$ is made of a random variable $x(t)$, with $x(0)$ being Gaussian of zero mean and constant covariance σ, namely $x(0) \sim N(0, \sigma)$. The observation error satisfies $x(t) - y(t) \sim N(0, r)$, and every time after the observation is made, $x(t)$ changes by $a\%$, where $|a| < 100$.

(a) Design a Kalman filter for the estimation of $x(t)$. . . . . (6p)

(b) For $a = 0$, express the covariance matrix $p(t) = E\{(x(t) - \hat{x}(t))^2\}$ in terms of σ, r. . . . . (7p)

(c) It is known that $p(t)$ converges to a positive semi-definite solution as t tends to ∞. Show the Kalman filter is asymptotically stable in the steady-state. . . (7p)

We can write down the Kalman filter as:

$$\hat{x}(t+1) = s\hat{x}(t) + sK(t)(y(t) - \hat{x}(t)),$$

where $s = 1 + 0.01a$, $K(t) = \frac{p(t)}{p(t)+r}$ and

$$p(t+1) = s^2 p(t) - s^2 \frac{p(t)^2}{p(t) + r}.$$

For $a = 0$, we have $p(t) = \frac{1}{\sigma + tr}$

When $a \neq 0$, we have in steady state $p = \frac{s^2 rp}{p+r}$. If $s < 1$, we have $p = 0$; if $s > 1$, we have $p = (s^2 - 1)r$. In both cases we have $s(1 - K) < 1$, thus asymptotically stable.

Good luck!