KTH Matematik

Exam March 18, 2011 in SF2832 Mathematical Systems Theory.

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Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

Read this before you start: 1. The problems are NOT ordered in terms of difficulty. 2. If the problem seems to be too complex (either in terms of calculation or abstraction), then it is likely that you have not found the best method yet.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

(*a*) Consider $\dot{x} = Ax,\ y = Cx$. If $x_0 \notin ker\ \Omega$, then $e^{At}x_0 \notin ker\ \Omega,\ \forall t \geq 0$. . . . (5p)

(*b*) Consider a n-dimensional time-invariant controllable system $\dot{x} = Ax + Bu$. Given $x_0,\ x_1$ and $T > 0$, there are infinitely many continuous control $u(t)$ that transfer the state of the system from $x(0) = x_0$ to $x(T) = x_1$. . . . . (5p)

(*c*) Suppose A is a stable matrix. For any positive definite matrix P, $-(A^T P + PA)$ is at least positive semi-definite. . . . . (5p)

(*d*) Suppose (A, b, c) is a minimal realization of a SISO transfer function $r(s) = \frac{n(s)}{d(s)}$, then $(c, A + bF)$ will still be observable for any F if and only if $n(s)$ is constant. (5p)

2. Consider a SISO system:

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned}$$

where

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & -A_1 \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ b_1 \end{bmatrix},\ c = \begin{bmatrix} c_1 & 0 & c_3 & 0 \end{bmatrix},$$

and the 2×2 matrix A_1 is such that $A_1^2 = 0$.

(a) Find the state transition matrix e^{At}, namely, express e^{At} as a matrix polynomial in terms of $A_1,\ A_2$. .... (6p)

(b) Suppose (A_1, b_1) is controllable and is already in controllable (reachable) canonical form, and $A_2 = [b_1, A_1 b_1]$. Show (A, b) is controllable........ (8p)

(c) Under the same assumptions as in (b), find conditions on $c_1,\ c_3$ such that under any pole placement, $(c, A + bF)$ is still observable. .... (6p)

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{\beta}{(s+1)(s+2)} \\ \frac{1}{s+1} & \frac{\gamma}{(s+1)(s+2)} \end{bmatrix},$$

where $\beta,\ \gamma$ are constants.

(a) Find the standard reachable realization........ (10p)

(b) Can the realization in (a) also be observable and why? .... (5p)

(c) For the case $\beta = \gamma = 0$, find a minimal realization of $R(s)$........ (5p)

4. In Chapter 3 we derived a minimum energy control for transfering the system from one state to another state. It is intuitive that the shorter time it takes to reach a given state the more energy the control spends. In this problem we show mathematically this is true.

Consider a controllable system

$$\dot{x} = Ax + Bu.$$

Suppose we want to transfer an arbitrary x_0 at $t = 0$ to the origin at $t = t_1$. It is proven that

$$\hat{u} = -B^T e^{A^T(t_1 - t)} W^{-1}(0, t_1) e^{At_1} x_0$$

where W is the reachability Gramian, is a feasible control that further minimizes among all feasible controls

$$J(u) = \int_0^{t_1} u^T(s) u(s) ds.$$

We denote $J(\hat{u}) = x_0^T L(t_1) x_0$.

(a) Show $W(0, t)$ satisfies $AW + WA^T + BB^T = e^{At} BB^T e^{A^T t}$. .... (4p)

(b) Show that for any $x_0 \neq 0$, $x_0^T L(t_2) x_0 < x_0^T L(t_1) x_0,\ t_2 > t_1$. (Hint: for a nonsingular matrix $M(t)$, $\frac{d}{dt}(M^{-1}(t)) = -M^{-1}\dot{M}M^{-1}$.)........ (8p)

(c) It is obvious that $\lim_{t_1 \to \infty} L(t_1) = 0$ if A is a stable matrix. What is $\lim_{t_1 \to \infty} L(t_1)$ if $-A$ is a stable matrix? (Hint: L would be determined if L^{-1} is.) .... (8p)

5. (*a*) Let x be the outcome of a random variable with distribution $N(0, \alpha^2)$ (i.e., $E\{x\} = 0,\ E\{x^2\} = \alpha^2$). We would like to determine the value of x by the following set of noisy measurements

$$y(t) = x + tw(t) \text{ for } t = 0, 1, \ldots, n, \ldots$$

where $w(t) \in N(0, \sigma^2)$ are independent of each other and of x. Let $\hat{x}_t = E^{H_{t-1}(y)}x$, and $P(t) = E\{(x - \hat{x}_t)^2\}$. Show $P(t+1) < P(t)$ and discuss what happens to $P(t)$ as $t \to \infty$. .... (10p)

(*b*) Consider $\dot{x} = Ax, y = Cx$. Show that the system is observable if and only if the only solution that satisfies $Cx(t) \equiv 0$ $(\forall t \geq 0)$ is $x(t) \equiv 0$. .... (10p)

Good luck!