KTH Matematik

Exam March 12, 2012 in SF2832 Mathematical Systems Theory.

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Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

Read this before you start: 1. The problems are NOT ordered in terms of difficulty. 2. If the problem seems to be too complex (either in terms of calculation or abstraction), then it is likely that you have not found the best method yet.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

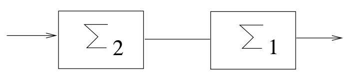
(a) Consider $\dot{x} = Ax$ where $x \in R^3$. If $\|x(t)\|^2 = \|x(0)\|^2$ for any $t > 0$ and $x(0)$, then A must have at least one eigenvalue equal to zero. . . . . (5p)

(b) Consider $\dot{x} = Ax,\ y = Cx$, where $x \in R^n$. If (C, A) is observable then $x(t)$ can be determined instantaneously from the output $y(t)$ and its derivatives up to $n - 1$ order. . . . . (5p)

(c) Given a minimal realization (A, B, C), $(A + BLC, B, C)$ is also minimal with any L. . . . . (5p)

(d) Consider the optimal control problem for $\dot{x} = Ax + Bu$: $\min_u \int_0^\infty (x^T Qx + u^T Ru)dt$, where $Q \geq 0$ and $R > 0$. If no symmetric solution P to the corresponding algebraic Riccati equation is positive definite, then the optimal control does not exist. . . . . (5p)

2. Consider a cascade connection of two systems:



where $\Sigma_i = (A_i, B_i, C_i), i = 1, 2$, and $dim(A_1) = n_1,\ dim(A_2) = n_2$.

(a) Determine the state space model for the overall cascaded system . . . . (6p) (*b*) Suppose $rank(B_2) = n_2$. Show the overall system is controllable if and only if (A_1, B_1C_2) is controllable. . . . . (8p)

(*c*) Now suppose the number of inputs is strictly less than n_2, but (A_2, B_2) is still controllable. Use an as simple as possible counter example to show that in this case (A_1, B_1C_2) being controllable does not necessarily imply that the overall system is controllable. . . . . (6p)

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{\beta}{s^2+2s+1} \\ \frac{1}{s+1} & \frac{\gamma}{s^2+2s+1} \end{bmatrix},$$

where $\beta,\ \gamma$ are nonzero constants.

(*a*) Find the standard observable realization. . . . . (10p)

(*b*) Can the realization in (a) also be controllable and why? . . . . (5p)

(*c*) For the case $\beta = \gamma$, find a minimal realization of $R(s)$. (5p)

4. In Chapter 3 we derived a minimum energy control for transferring the system from one state to another state. It is intuitive that the shorter time it takes to reach a given state the more energy the control spends. In this problem we show mathematically this is true.

Consider a controllable system

$$\dot{x} = Ax + Bu.$$

Suppose we want to transfer an arbitrary x_0 at $t = 0$ to the origin at $t = t_1$. It is proven that

$$\hat{u} = -B^T e^{A^T(t_1-t)} W^{-1}(0, t_1) e^{At_1} x_0$$

where W is the reachability Gramian, is a feasible control that further minimizes among all feasible controls

$$J(u) = \int_0^{t_1} u^T(s)u(s)ds.$$

We denote $J(\hat{u}) = x_0^T L(t_1) x_0$.

(*a*) Show $W(0, t)$ satisfies $AW + WA^T + BB^T = e^{At}BB^Te^{A^Tt}$. . . . . (4p)

(*b*) Show that for any $x_0 \neq 0$, $x_0^T L(t_2) x_0 < x_0^T L(t_1) x_0$, $t_2 > t_1$. (Hint: for a nonsingular matrix $M(t)$, $\frac{d}{dt}(M^{-1}(t)) = -M^{-1}\dot{M}M^{-1}$.) . . . . (8p)

(*c*) It is obvious that $\lim_{t_1\to\infty} L(t_1) = 0$ if A is a stable matrix. What is $\lim_{t_1\to\infty} L(t_1)$ if $-A$ is a stable matrix? . . . . (8p)

5. (*a*) This problem is related to the motion of a particle. Given a nonzero $b \in R^3$ (translational motion), show that for almost all skew symmetric matrices A ($A^T = -A$, rotation), (A, b) is controllable (8p). Give a geometric interpretation to those skew symmetric A such that (A, b) is not controllable (4p). .... (12p)

(*b*) Let z be the outcome of a random variable with distribution $N(0, \alpha^2)$ (i.e., $E\{z\} = 0,\ E\{z^2\} = \alpha^2$). We would like to determine the value of z by a set of noisy measurements

$$y(t) = z + w(t) \text{ for } t = 0, 1, \ldots, n-1$$

where $w(t) \in N(0, \sigma^2)$ are independent of each other and of z.
Determine $P(n) = E\{(z - \hat{z}_n)^2\}$ as a function of α, σ, n, where $\hat{z}_n$ is the optimal estimation of z based on measurements up to time instant $n - 1$. .... (8p)

Good luck!