# SF2832
# Solutions to 2012 Exam

(For reference only)

1.

a) True. $\|x(t)\|^2 = x_0^T e^{A^T t} e^{At} x_0 = x_0^T x_0 \quad \forall x_0$

$\Rightarrow \quad e^{A^T t} e^{At} = I \quad \Rightarrow \quad A^T = -A$

Since $\det(A) = \det(A^T) = \det(-A) = (-1)^3 \det A$

$\Rightarrow \quad \det(A) = 0 \quad \Rightarrow \quad \lambda_1 = 0.$

b) True. Since $y^{(i)}(t) = CA^i x(t)$

c) True. Since feedback does not change controllability and output feedback does not change observability.

d) False. Example: $Q = 0$, $A = -I$, which implies $P = 0$.

2.

a) $\dot{x}_1 = A_1 x_1 + B_1 C_2 x_2$

$\dot{x}_2 = A_2 x_2 + B_2 u$

$y = C_1 x_1$

b) Since $\operatorname{rank}(B_2) = n_2$, we can assume no. of inputs $= n_2$.
$\Rightarrow B_2$ nonsingular. Since $u = Kv$ where $K$ nonsingular does not change controllability, we can let $K = B_2^{-1}$. Since feedback does not change controllability, we can let $u = B_2^{-1}(-A_2 x_2 + v)$.

$\Rightarrow \quad A = \begin{bmatrix} A_1 & B_1 C_2 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I \end{bmatrix}.$ The result follows.

2.

c) $\dot{x}_1 = x_{22}$

$\dot{x}_{21} = x_{22}$ ( $B_1 = 1$ , $C_2 = [0 \ 1]$ )

$\dot{x}_{22} = u$

3.

a) $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ -1 & \beta \\ -1 & \gamma \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

b) No. Since $\delta = 3$ if $\beta \neq \gamma$ and $\delta = 2$ if $\beta = \gamma$

c) When $\beta = \gamma$, $Y_1(s) = Y_2(s)$, which implies we only need to find a minimum realization from a) by considering only $y_1$ (or $y_2$) as the output. One can see easily that $x_2$, $x_4$ has no influence on $y_1$, thus

$\dot{x}_1 = x_3 + u_1$

$\dot{x}_3 = -x_1 - 2x_3 - u_1 + \beta u_2$

$y = x_1$

is a realization that happens to be already minimal!

$\Rightarrow$ $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -1 & \beta \end{bmatrix}$, $C = [1 \ 0]$

4.

a) Use the same trick as on p. 33 of the Compendium, where we let $Q = BB^T$ and replace $A$ by $A^T$.

4.

b) We can show $\dot{\mathcal{L}} = -\mathcal{L}\, e^{(-At)} B B^T e^{(-A^T t)} \mathcal{L}$

or $\dot{\mathcal{L}^{-1}} = e^{-At} B B^T e^{-A^T t}$ (this is easier to calculate but less intuitive).

where, $\mathcal{L} = e^{A^T t} W^{-1}(0,t) e^{At}$

c) from the equality in a), we have

$$-A\mathcal{L}^{-1} - \mathcal{L}^{-1}A^T - e^{-At} B B^T e^{-A^T t} = -BB^T$$

and the result follows.

5 (5.b) is omitted)

a). Let $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$, $\|b\| \neq 0$, $A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$,

and denote $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

Method 1: Since $Ab = a \otimes b$, $A^2 b = a \otimes (a \otimes b)$.

as long as $a$ is not parallel or perpendicular to $b$, $[b \;\; Ab \;\; A^2 b]$ is nonsingular.

method 2: We let $\hat{b} = b/\|b\|$ and use $\hat{b}$ as the first (2nd, or third) column of an orthonormal Transform matrix $T$. Then in the new coordinates

$\hat{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\bar{A} = \begin{bmatrix} 0 & -\bar{a}_3 & \bar{a}_2 \\ \bar{a}_3 & 0 & -\bar{a}_1 \\ -\bar{a}_2 & \bar{a}_1 & 0 \end{bmatrix}$

Computing $[\hat{b} \;\; \bar{A}\hat{b} \;\; \bar{A}^2\hat{b}]$ the conclusion follows.