KTH Matematik

Solution to Exam April 8, 2015 in SF2832 Mathematical Systems Theory.

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Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

(a) Consider $\dot{x} = A(t)x$ where $x \in R^n$ and $A(t)$ is continuous on $(-\infty, \infty)$. If $\|x(t)\|^2 = \|x(0)\|^2$ for any $t \geq 0$ and $x(0)$, then $A(t) + A^T(t) = 0 \ \forall t \geq 0$. . (5p)

Answer: True. $\|x(t)\|^2 = \|x(0)\|^2$ implies that $\frac{d}{dt}\|x(t)\|^2 = 0$.

(b) If (A, B) is controllable, then for any $\bar{B}$ such that $\bar{B}\bar{B}^T = BB^T + Q$, where Q is positive semi-definite, $(A, \bar{B})$ is also controllable. . . . . (5p)

Answer: True. Easily seen from the controllability Gramian.

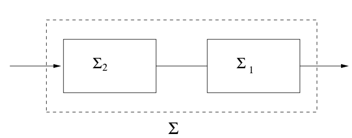
(c) Given a strictly proper rational matrix function $R(s)$, if the dimension of its standard reachable realization is equal to that of its standard observable realization, then that dimension must be equal to the McMillan degree of $R(s)$. (5p)

Answer: Obviously false.

(d) If (A, B, C) is not a minimal realization, then the algebraic Riccati equation $A^TP + PA - PBB^TP + C^TC = 0$ does not have any positive definite solution P. . . . . (5p)

Answer: False. For example, A is a stable matrix, (C, A) is observable, $B = 0$.

2. Consider system Σ resulting from a cascade connection of two systems:



where $\Sigma_i = (A_i, B_i, C_i), i = 1, 2,$

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix},\ C_2 = \begin{bmatrix} q & 1 \end{bmatrix}$$

where q is a real constant.

(a) Determine the state space model for the overall cascaded system Σ. ..... (5p)
Answer: omitted.

(b) For what q is Σ observable? ..... (5p)
Answer: $q \neq \pm 1$.

(c) For what q is Σ controllable? ..... (5p)
Answer: $q \neq \frac{1}{2}$

(d) Suppose q is such that Σ is both controllable and observable. When we apply a control to force $y(t) = 0\ \forall t \geq 0$ (this implies that $x(0)$ must be chosen such that $y(0) = 0$), what is the further constraint on q such that the corresponding $x(t)$ always tend to zero as t tends to ∞? ($y(t)$ and $x(t)$ are the output and the state of Σ respectively.) ..... (5p)
Answer: $q > 0$.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{s} & \frac{1}{s} \\ \frac{1}{s^2+as} & \frac{1}{s^2+as} \end{bmatrix},$$

where $a,\ \gamma$ are nonzero constants.

(a) Find the standard observable realization. ..... (8p)
Answer: omitted.

(b) Compute the McMillan degree of $R(s)$. ..... (4p)
Answer: $\delta(R) = 3$ if $\gamma \neq 1$, otherwise 2.

(c) For the case $\gamma \neq 1$, find a minimal realization of $R(s)$. ..... (8p)
Answer: Let $\bar{u}_1 = \gamma u_1 + u_2,\ \bar{u}_2 = u_1 + u_2$, we can easily find realization for $y_1 = \frac{1}{s}\bar{u}_1$, and $y_2 = \frac{1}{s^2+as}\bar{u}_2$.

4. Consider the optimal control problem

$$\begin{aligned} \min_u J &= \int_0^\infty (\epsilon^2 y^2 + u^2)dt \\ \text{s.t.} & \\ \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0, \end{aligned}$$

where, $\epsilon \neq 0$, and

$$A = \begin{bmatrix} a_1 & 1 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

(*a*) Show for $a_1 = 0$, the associated algebraic Riccati equation (ARE) does not have a positive definite solution. . . . . (6p)

Answer: By solving the ARE we can easily see this.

(*b*) Let $P(\epsilon)$ denote the symmetric solution to the ARE. Show that $\lim_{\epsilon \to 0} P(\epsilon)$ is positive definite if and only if $a_1 > 0$ and $a_2 > 0$. . . . . (8p)

Answer: P satisfies $A^T P + PA - PBB^T P = 0$. When $a_1 > 0, \ a_2 > 0$, $P = 0$ can not be the solution. The rest follows from this Lyapunov equation: $W(-A)^T + (-A)W + BB^T = 0$.

(*c*) Show when $\lim_{\epsilon \to 0} P(\epsilon) > 0$, $\lim_{\epsilon \to 0}(A - BB^T P(\epsilon))$ has eigenvalues $\{-a_1, -a_2\}$. (6p)

Answer: Since $P(A - BB^T P) = -A^T P, \ P(A - BB^T P)P^{-1} = -A^T$, and the conclusion follows.

5. (*a*) In this problem we investigate the relative degree of single-input-single-output systems (SISO). Consider the state space representation

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

and its corresponding transfer function

$$G(s) = C(sI - A)^{-1}B = k\frac{s^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

where $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times 1}$, $C \in \mathbf{R}^{1 \times n}$, and $k \neq 0$. The relative degree $r = n - m$ is the excess degree of the denominator compared to the numerator. Prove that

$$CA^k B = 0, \quad k = 0, 1, \ldots, r-2 \quad \text{and} \quad CA^{r-1}B \neq 0 \quad \text{(6p)}$$

and $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix}$ has full row rank (4p). . . . . (10p)

Answer: By Laurent expansion we can show that $CA^k B = 0, \quad k = 0, 1, \ldots, r-2$ and $CA^{r-1}B \neq 0$ ($R_i = CA^{i-1}B$). We can show the second statement by contradiction: assume $\sum_{i=1}^{r} \alpha_i CA^{i-1} = 0$. Multiply both sides of the equality by $A^k B, \ k = 0, 1, \ldots$ repeatedly, we conclude that $\alpha_i = 0, \ i = r, \ldots, 1$.

(*b*) Consider a controllable system

$$\dot{x} = Ax + Bu.$$

Show that for any $t_1 > 0$, $u = -B^T W^{-1}(t_1)x$ asymptotically stabilizes the system, where

$$W(t_1) = \int_0^{t_1} e^{-At} B B^T e^{-A^T t} dt.$$

Hint: use results and ideas in Section 4.3 of the compendium. .... (10p)

Answer: Let $Q = e^{-At} B B^T e^{-A^T t}$, then $\frac{d}{dt} Q = -AQ - QA^T$. We have $e^{-At_1} B B^T e^{-A^T t_1} - BB^T = -AW - WA^T$. Thus $(A - BB^T W^{-1})W + W(A - BB^T W^{-1})^T = -BB^T - e^{-At_1} BB^T e^{-A^T t_1}$, and the conclusion follows.

Good luck!