KTH Matematik

Solution to Exam January 15, 2015, SF2832 Mathematical Systems Theory.

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Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

(a) Consider $\dot{x} = A(t)x,\ x \in R^n$. $\Phi(t,s) = \Phi(t-s)$ if and only if $A(t)$ is a constant matrix, where $\Phi(t,s)$ is the state transition matrix. .... (5p)

Answer: True. "only if": Let $\tau = t - s$. Since $\Phi(t,s) = \Phi(t-s,0)$, then $A(t)\Phi = \frac{\partial \Phi}{\partial t} = \frac{\partial \Phi}{\partial \tau}\frac{\partial \tau}{\partial t} = A(t-s)\Phi$. Let $s = t$ we have $A(t) = A(0)$.

(b) Consider the motion of a rotating particle: $\dot{x} = Ax$, where $x \in R^3$ and $A^T = -A$, $A \neq 0$. Then we can always design a single output $y = cx$ (i.e. $y \in R$) such that (c, A) is observable. .... (5p)

Answer: True. Since $\|x(t)\|^2 = x_0^T e^{-At} e^{At} x_0 = \|x_0\|^2$, all three eigenvalues are on the imaginary axis. Since $A \neq 0$, we have $\lambda_1 = 0, \lambda_2 = j\omega, \lambda_3 = -j\omega$. Thus A is cyclic, which implies $\exists\ c$ such that (A^T, c^T) is controllable.

(c) Consider a controllable system

$$\dot{x} = Ax + Bu$$
$$y = Cx,$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, and $C \neq 0$. Then, we can always find $u = Kx + v$ such that $(C, A + BK)$ is observable. .... (5p)

Answer: True. $C \neq 0$ implies at least one row $c_k \neq 0$. Let K_1 be such that $A + BK_1$ has distinct eigenvalues, if $(c_k, A + BK_1)$ is observable, then $K = K_1$. Otherwise let u_0 be such that $(A + BK_1, Bu_0)$ is controllable and choose k_2 such that the eigenvalues of $A + BK_1 + Bu_0k_2$ do not coincide with the zeros of system $(A + BK_1, Bu_0, c_k)$, which gives $K = K_1 + u_0k_2$.

(d) If A is a stable matrix and (C, A) is observable, then the algebraic Riccati equation $A^TP + PA - PBB^TP + C^TC = 0$ always has a positive definite solution P, no matter what B is. .... (5p)

Answer: True. A being a stable matrix implies $u = 0$ is a feasible control. Thus ARE has a semi-definite solution P. Using exactly the same argument as in the compendium, we see that observability implies P is further positive definite.

2. Consider :

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & -a_1 & 0 \\ a_2 & 0 & 1 \\ 0 & 0 & a_3 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ c = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},$$

and a_1, a_2, a_3 are real numbers satisfying $a_2 \neq 1, \ a_1 a_2 > 0$.

(*a*) Find the state transition matrix e^{At}. .... (5p)

Answer:

$$\begin{bmatrix} \cos(\sqrt{a_1 a_2}t) & -\sqrt{\frac{a_1}{a_2}}\sin(\sqrt{a_1 a_2}t) & \frac{a_1}{a_1 a_2 + a_3^2}\left[\cos(\sqrt{a_1 a_2}t) + \frac{a_3}{\sqrt{a_1 a_2}}\sin(\sqrt{a_1 a_2}t) - e^{a_3 t}\right] \\ \sqrt{\frac{a_2}{a_1}}\sin(\sqrt{a_1 a_2}t) & \cos(\sqrt{a_1 a_2}t) & \frac{1}{a_1 a_2 + a_3^2}\left[-a_3\cos(\sqrt{a_1 a_2}t) + \sqrt{a_1 a_2}\sin(\sqrt{a_1 a_2}t) + a_3 e^{a_3 t}\right] \\ 0 & 0 & e^{a_3 t} \end{bmatrix}$$

(*b*) Propose a set of three desired eigenvalues for $A + bk$ such that $(c, A + bk)$ is always observable, and show those eigenvalues can actually be placed by some k (you do not have to give such a row vector k explicitly). .... (6p)

Answer: We can first verify that the system is controllable, thus arbitrary pole placement is possible. Then we can compute that the zeros of the system are defined by $s^2 + s + a_1(a_2 - 1) = 0$. Thus if we place the poles at, for example, $s_1 = s_2 = s_3 = 0$, the closed-loop system will be always observable.

(*c*) Find a linear feedback control $u(t) = kx$ such that whenever $cx(0) = 0$, then $y(t) = 0$ for all $t \geq 0$. .... (4p)

Answer: $u = -a_2 x_1 + a_1 x_2 - (1 + a_3) x_3$.

(*d*) What happens to $x(t)$ as $t \to \infty$ when the control designed in (c) is applied and the initial condition is such that $cx(0) = 0$? .... (5p)

Answer: If $a_1(a_2 - 1) > 0$, $x(t)$ converges to 0; if $a_1(a_2 - 1) = 0$, $x(t)$ converges; if $a_1(a_2 - 1) < 0$, $x(t)$ diverges.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{(s+1)^2} & \frac{1}{(s+1)^2} \\ \frac{1}{s^2+s} & \frac{1}{s^2+s} \end{bmatrix},$$

where γ is a constant.

(*a*) Find the standard reachable realization. .... (6p)

Answer: Omitted.

(*b*) Compute the McMillan degree of $R(s)$. .... (5p)

Answer: $\delta(R) = 4$ if $\gamma \neq 1$ otherwise $\delta(R) = 3$.

(c) For the case $\gamma \neq 1$, find a minimal realization of $R(s)$ and verify your answer if you use Kalman decomposition. . . . . (9p)

Answer: The answer is not unique, but the dimension must be 4. One way to solve: let $u_1 = \bar{u}_1 - \bar{u}_2,\ u_2 = -\bar{u}_1 + \gamma\bar{u}_2$, then we have

$$y = \begin{pmatrix} \frac{\gamma-1}{(s+1)^2} & 0 \\ 0 & \frac{(\gamma-1)}{s^2+s} \end{pmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix},$$

for which we can easily find a minimal realization, then replace $\bar{u}_1, \bar{u}_2$ by expressions of u_1 and u_2.

4. Consider the optimal control problem

$$\min_u J = \int_{t_0}^{t_1} ((x_1 - x_2)^2 + \epsilon^2 u^2)dt$$
$$\text{s.t.}$$
$$\dot{x} = Ax + Bu$$
$$x(t_0) = x_0,$$

where, $t_1 > t_0 \geq 0$, $\epsilon > 0$, and

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_1 \end{bmatrix},$$

where a_1, a_2, b_1 are real constants.

(a) For what a_1, a_2, b_1 we have $P(t_1 - t) > 0,\ \forall t \in [t_0, t_1)$? Here $P(t_1 - t)$ is associated with the optimal cost $x_0^T P(t_1 - t)x_0$ if the initial time is taken to be t. Note that you do not need to find $P(t_1 - t)$ explicitly. . . . . (6p)

Answer: Let $Q = C^T C = [1 \ \ -1]^T[1 \ \ -1]$. By contradiction, we can show that $P(t_1 - t) > 0,\ \forall t < t_1$ iff (C, A) is observable, since $x_t^T P(t_1 - t)x_t$ is the optimal cost for $\int_t^{t_1}(x^T Qx + \epsilon^2 u^2)ds$ with $x(t) = x_t$. If $P(t_1 - t)$ is not p.d., then $\exists\ \bar{x} \neq 0$ such that $\bar{x}^T P(t_1 - t)\bar{x} = 0$, which implies $u = 0$ and $Cx = 0$ on $[t, t_1]$. The latter contradicts with observability.

(C, A) is observable if $a_1 \neq a_2$.

(b) For the case $P(t_1 - t) > 0,\ \forall t \in [t_0, t_1)$, discuss when $P_\epsilon := \lim_{t_1 - t \to \infty} P(t_1 - t)$ exists and is positive definite . . . . (6p)

Answer: When (A, B) is controllable $(a_1 \neq a_2,\ b_1 \neq 0)$, $P_\epsilon > 0$. When the system is not controllable (but still need to be observable), $P_\epsilon > 0$ if $a_1 < 0,\ a_2 < 0$ and $a_1 \neq a_2$.

(c) Let $a_1 \neq 0;\ a_2 = 0,\ b_1 = 1$, and $K_\epsilon = \epsilon^{-2}B^T P_\epsilon$, what are the eigenvalues of $A - BK_\epsilon$ as $\epsilon \to 0$? (Hint: coordinate change may help) . . . . (8p)

Answer: Let $\bar{x}_1 = x_1 - x_2,\ \bar{x}_2 = x_2,\ P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$, then

$$\dot{\bar{x}}_1 = a_1\bar{x}_1 + a_1\bar{x}_2$$
$$\dot{\bar{x}}_2 = u.$$

Solve the corresponding ARE: $p_3 \approx \epsilon\sqrt{2|a_1|\epsilon}$, $p_2 = p_3 + sign(a_1)\epsilon$. Then we can see both eigenvalues tend to $-\infty$.

You may also argue as follows (if you do not have time to solve the ARE) and receive some points: in the new coordinates $\bar{x}_1^2 = x^T Qx$, and we control $\bar{x}_1$ via $\bar{x}_2$. So to drive $\bar{x}_1$ to 0 as fast as possible requires to drive $\bar{x}_2$ to 0 as fast as possible, thus, ...

5. (*a*) Consider a one-dimensional system

$$x(t+1) = ax(t)$$
$$y(t) = x(t) + w(t),$$

where $a \neq 0$,, both $x(0)$ and $w(t)$ are Gaussian with zero mean and covariances p_0 and σ respectively. respectively.

(*i*) Design a Kalman filter $\hat{x}(t)$ for $x(t)$. . . . . (3p)
Answer: $k(t) = p(t)(p(t) + \sigma)^{-1}$, $p(t+1) = \frac{\sigma a^2 p(t)}{p(t)+\sigma}$.

(*ii*) Express the covariance matrix $p(t) = E\{(x(t)-\hat{x}(t))^2\}$ in terms of t, a, p_0, σ. (4p)
Answer: $p^{-1}(t+1) = a^{-2}(p^{-1}(t)+\sigma^{-1})$. Then, $\frac{1}{p(t)} = a^{-2t}p_0^{-1}+\sigma^{-1}\sum_{i=1}^{t} a^{-2i}$.

(*iii*) Show $|a - ak(t)| \leq 1$ as $t \to \infty$ (where $k(t)$ is the Kalman gain)? . . . . (3p)
Answer: Let $p^{-1} = a^{-2}(p^{-1} + \sigma^{-1})$ or $p = \frac{\sigma a^2 p}{p+\sigma}$ be the steady state $p(t)$, then $p = 0$ if $|a| \leq 1$ and $p = \sigma(a^2 - 1)$ otherwise. The conclusion then follows.

(*b*) Consider a minimal system $R(s) = C(sI - A)^{-1}B$. In the literature, such a system is called positive real if there exists a positive definite matrix S and a matrix L such that

$$A^T S + SA = -L^T L$$
$$SB = C^T.$$

Assume now the positive definite solution P to the ARE

$$A^T P + PA - PBB^T P + C^T C = 0$$

satisfies also $PB = C^T$. Show

(*i*) $R(s) = C(sI - \bar{A})^{-1}B$ where $\bar{A} = A - BB^T P$ is positive real. . . . . (5p)
Answer: Since $\bar{A}^T P + P\bar{A} = -PBB^T P - C^T C = -2C^T C$, let $S = P$ and $L = \sqrt{2}C$, we are done.

(*ii*) $(C, \bar{A})$ is observable.
Answer: Let $\dot{x} = (A - BB^T P)x = Ax - BCx = Ax - By$, $y = Cx$, since output feedback does not change observability, $(C, \bar{A})$ is observable. . (5p)

Good luck!