KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
08.00-13.00, April 11, 2017

Examiner: Xiaoming Hu, tel. 790 7180.

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

Read this before you start: 1. The problems are NOT ordered in terms of difficulty. 2. If the problem seems to be too complex (either in terms of calculation or abstraction), then it is likely that you have not found the best method yet.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

(*a*) Consider an n-dimensional *time-varying* system $\dot{x} = A(t)x$, where $A(t)$ is continuous. If $A^T(t) = -A(t)$ $\forall t \in R$, then $\Phi^T(t,s) = \Phi^{-1}(t,s)$, where $\Phi(t,s)$ is the state transition matrix. . . . . (5p)

Answer: True, since $(\frac{\partial \Phi(t,s)}{\partial s})^T = A(s)\Phi(t,s)^T$, thus $\Phi(t,s)^T = \Phi(s,t) = \Phi^{-1}(t,s)$.

(*b*) Consider $\dot{x} = Ax, y = cx$, where $x \in R^n,\ y \in R$. If $rank\ A < n-1$, then (c, A) is never observable. . . . . (5p)

Answer: True. (c, A) observerble is the same as that $\dot{z} = A^T z + c^T v$ is controllable. Putting the system into the canonical controllable form, we know that A^T must have rank at least $n-1$.

(*c*) Consider $\dot{x} = Ax + bu, y = cx$, where $x \in R^n,\ u \in R,\ y \in R$. If $c(sI-A)^{-1}b = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$, then $(c, A+bk)$ is observable for any k. . . . . (5p)

Answer: True. We can show this by realizing the transfer function in standard controllable realization.

(*d*) Consider the Riccati differential equation:

$$\begin{aligned} \dot{P}(t) &= -A^T P(t) - P(t)A + P(t)BB^T P(t) - C^T C \\ P(t_1) &= P_1, \end{aligned}$$

where C is a $p \times n$ matrix and $p < n$. If P_1 is only positive semidefinite ($det\ P_1 = 0$), then $det\ P(t) = 0$ for any $t < t_1$. . . . . (5p)

Answer: False, for example, when the system is minimal.

2. Consider :

$$\begin{aligned}\dot{x} &= Ax + bu\\ y &= cx\end{aligned}$$

where

$$A = \begin{bmatrix} 0 & -a_1 & 0 \\ a_1 & 0 & -a_2 \\ 0 & a_2 & 0 \end{bmatrix},\ b = \begin{bmatrix}1\\0\\0\end{bmatrix},\ c = \begin{bmatrix}1 & 0 & 0\end{bmatrix},$$

and $a_1 > 0,\ a_2 > 0$ and $a_1^2 + a_2^2 = 1$.

(*a*) Find the state transition matrix e^{At}. .... (6p)
Answer: $e^{At} = I + A\sin(t) + A^2(1-\cos(t))$.

(*b*) When can the system be asymptotically stabilized by a control $u = Kx$? (4p)
Answer: The system is always controllable.

(*c*) For any given initial state $[0\ x_2(0)\ x_3(0)]^T$ where $x_2(0)^2 + x_3(0)^2 \neq 0$, find an open-loop control $u(t)$ ($u(t)$ is expressed as a function of the initial state and time t) such that $x_1(t) = 0\ \forall t \geq 0$. .... (6p)
Answer: We can see easily that $u = a_1x_2(t)$ will keep $x_1(t)$ zero, but it is not in open-loop. But with the control and the initial condition, we have $\dot{x}_2 = -a_2x_3, \dot{x}_3 = a_2x_2$, thus $x_2(t) = x_2(0)\cos(a_2t) - x_3(0)\sin(a_2t)$, then $u = a_1(x_2(0)\cos(a_2t) - x_3(0)\sin(a_2t))$.

(*d*) If we plot $x_2(t),\ x_3(t)$ obtained in (c) on the (x_2, x_3)-plane for $0 \leq t < \infty$, how does the trajectory look? .... (4p)
Answer: A circle centered at the origin and with radius $\sqrt{x_2(0)^2 + x_3(0)^2}$.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\alpha}{s+1} & \frac{1}{s+\beta} \\ \frac{1}{s+1} & \frac{1}{s+\beta} \end{bmatrix},$$

where α, β are real nonzero constants.

(*a*) Determine the McMillan degree of $R(s)$. .... (8p)
Answer: If $\alpha \neq 1$ or $\beta \neq 1$, $\delta(R) = 2$. If $\alpha = \beta = 1$, $\delta(R) = 1$.

(*b*) Find the standard reachable realization. .... (8p)
Answer: omitted.

(*c*) When is the realization in (b) also observable? .... (4p)
Answer: $\alpha \neq 1$ and $\beta = 1$.

4. Consider the optimal control problem

$$\min_u J = \int_0^{t_1} u^T u dt + x(t_1)^T S x(t_1) \quad \text{s.t.} \quad \dot{x} = Ax + Bu,\ x(0) = x_0,$$

where (A, B) is controllable and S is positive definite.

Let $u = -B^T P(t_1 - t)x$ denote the optimal control.

(a) Solve the Riccati equation to obtain $P(t_1 - t)$. (**Hint:** to determine P is the same as determining P^{-1} if P is invertible) .... (10p)

Answer: By using the adjoint system, we have $Y = exp(A^T(t_1 - t))S$, $X = exp(-A(t_1 - t)) + \int_0^{t_1-t} exp(-As)BB^T exp(-A^T s)dsexp(A^T(t_1 - t))S$.

$P^{-1} = XY^{-1} = exp(-A(t_1-t))S^{-1}exp(-A^T(t_1-t))+\int_0^{t_1-t} exp(-As)BB^T exp(-A^T s)ds.$

(b) Compute $\lim_{t_1-t\to\infty} P(t_1 - t)$ for the case A is a stable matrix. .... (4p)

Answer: If A is stable, $P^{-1} \to \infty$

(c) What are the eigenvalues of $\lim_{t_1-t\to\infty}(A - BB^T P(t_1 - t))$ for the case $-A$ is a stable matrix? .... (6p)

Answer: If $-A$ is stable, $P^{-1} \to \int_0^\infty exp(-As)BB^T exp(-A^T s)ds$, which satisfies $-P^{-1}A^T - AP^{-1} + BB^T = 0$. Thus, $A - BB^T P = -P^{-1}A^T P$, which has same eigenvalues as $-A^T$ thus as $-A$.

5. (a) Let

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}.$$

Show

$$det\ \Phi(t, t_0) = e^{\int_{t_0}^t (a_{11}(s)+a_{22}(s))ds},$$

where $\Phi(t, t_0)$ is the state transition matrix. .... (5p)

Answer: Let $\Phi(t, t_0) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$, then $det\ \Phi(t, t_0) = \phi_{11}\phi_{22} - \phi_{12}\phi_{21}$. $\frac{d}{dt}(det\ \Phi(t, t_0)) = (a_{11} + a_{22})det\ \Phi(t, t_0)$, and $det\ \Phi(t_0, t_0) = 1$.

(b) Consider the algebraic Riccati equation

$$A^T P + PA - PBB^T P + C^T C = 0.$$

(1) Assume P is a real positive **semidefinite** solution. Show that $kerP$ is A-invariant (i.e, $\forall x \in kerP,\ Ax \in kerP$) and $kerP \subset kerC$. .... (4p)

Answer: Suppose $x \in KerP$. Multiplying both sides of the ARE by x:

$$PAx + C^T Cx = 0,$$

similarly $x^T C^T Cx = 0$. Therefore $x \in KerC$, which implies $kerP \subset kerC$. Furthermore, this leads to that $PAx = 0$. Thus $kerP$ is A-invariant.

(2) Show that if (C, A) is observable, then every positive semidefinite solution P is positive definite. **Hint:** use the conclusions in (1) .... (4p)

Answer: When (C, A) is observable, the only A-invariant subspace in $KerC$ (unobservable subspace) is $\{0\}$. Thus, $kerP = \{0\}$.

(c) All conclusions about Kalman filter still hold if we replace $\mathcal{E}\{w(t)w^T(t)\} = R > 0$ by $\mathcal{E}\{w(t)w^T(t)\} = R(t) > 0$. Namely allow the covariance matrix for the noise to be time-varying.

Now consider the problem of measuring some constant scalar quantity x. Suppose initially nothing is known about x (i.e. $P(0) = \infty$). Then at each time instance $t = 0, 1, \cdots, n$, $y(t)$, a measurement of x, is made with error covariance $r(t)$.

(1) Express the optimal estimation of x at t, $\hat{x}(t)$, which is based on measurements up to $t - 1$, by Kalman filter. . . . . (3p)
Answer: omitted.

(2) Write down the expression of $P(t)$ in the Kalman filter in terms of $r(i)$, $i = 0, 1, \cdots, t - 1$. . . . . (4p)
Answer: $P(t)^{-1} = \sum_{i=0}^{t-1} r(i)^{-1}$.

Good luck!