KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
14:00-19:00, January 8, 2019

Examiner: Xiaoming Hu, tel. 790 7180.

Allowed books: Course compendium by Anders Lindquist et. al, Exercise notes by Per Enqvist, your own hand-written class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course homepage.

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1. Determine if each of the following statements is true or false. You must justify your answers. All matrices involved are assumed to be constant matrices unless otherwise specified.

(a) Consider an n-dimensional time-varying system $\dot{x} = A(t)x$, where $A(t)$ is continuous and $A^T(t) = -A(t)\ \forall t \in R$. Then the solution $x(t)\ \forall t \geq t_0$ lies on a sphere with radius $r = |x_0|$, where $x(t_0) = x_0 \neq 0$. . . . . (5p)
Answer: True since $\|x(t)\|^2 = x_0^T \Phi^T(t,t_0)\Phi(t,t_0)x_0 = x_0^T\Phi(t_0,t)\Phi(t,t_0)x_= \|x_0\|^2$.

(b) Consider $\dot{x} = Ax$ where $x \in R^n$. If $x = 0$ is unstable, then A must have at least one eigenvalue with positive real part. . . . . (5p)

Answer: False. For example $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is unstable.

(c) Consider

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx, \end{aligned}$$

where $x \in R^n,\ y \in R^p$. Assume $x(0) = x_0 \notin ker\ \Omega$, then $y(t) = Ce^{At}x_0$ can not be identically zero over any time interval $[t_1, t_2]$, where $t_2 > t_1 \geq 0$. . . (5p)
Answer: True. $Ce^{At}x_0 \equiv 0$ at $[t_1,t_2]$ implies that $\Omega e^{At}x_0 \equiv 0$, thus $x_0 \in ker\ \Omega$ since $ker\ \Omega$ is A-invariant.

(d) If (C, A) is observable, then the algebraic Riccati equation $A^TP + PA - PBB^TP + C^TC = 0$ always has a positive definite solution P, no matter what B is. (5p)
Answer: False, for example when $B = 0$ and A is unstable.

2. Consider :

$$\dot{x} \quad = \quad Ax + bu$$

where

$$A = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \text{and } \alpha \text{ is constant.}$$

(a) Find the state transition matrix e^{At}. .... (6p)
Answer: We solve first $x_3(t) = e^{-t}x_{30}$, then plug it in to the second equation to obtain $x_2(t)$, and then plug $x_2(t)$ in to the first equation. The rest is omitted.

(b) When is the pole placement problem solvable? .... (3p)
Answer: When the system is controllable, namely $\alpha \neq -2$.

(c) Let $u = 0$. Find all solutions $x(t)$ that lie all the time on the plane $D = \{x \in R^3 : [0\ 1\ 0]x = 0\}$. .... (5p)
Answer: $x_1(t) = e^{\alpha t}x_{10}, x_2(t) = 0, x_3(t) = 0$.

(d) Find $u(t) = Kx$ that makes D invariant, i.e., $[0\ 1\ 0]x(t) = 0, \forall t > 0$ if $[0\ 1\ 0]x(0) = 0$. .... (3p)
Answer: $u = -x_3$.

(e) Do the solutions on D converge to the origin as $t \to \infty$ when the control designed in (d) is applied? .... (3p)
Answer: When $\alpha < 0$.

3. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{s} & \frac{s+2}{s(s+1)} \\ \frac{1}{s} & \frac{s+2}{s(s+1)} \end{bmatrix},$$

where γ is a nonzero constant.

(a) Find the standard reachable realization. .... (7p)

(b) Compute the McMillan degree of $R(s)$. .... (6p)

(c) For the case $\gamma = 1$, find a minimal realization of $R(s)$. .... (7p)

Answer: See Exam of Jan. 2014.

4. In Chapter 3 we derived a minimum energy control for transferring the system from one state to another state. It is intuitive that the shorter time it takes to reach a given state the more energy the control spends. In this problem we show that mathematically this is true.

Consider a controllable system

$$\dot{x} = Ax + Bu.$$

Suppose we want to transfer an arbitrary x_0 at $t = 0$ to the origin at $t = t_1$. It is proven that

$$\hat{u} = -B^T e^{A^T(t_1-t)} W^{-1}(0, t_1) e^{At_1} x_0$$

where W is the reachability Gramian, is a feasible control that further minimizes among all feasible controls $J(u) = \int_0^{t_1} u^T(s)u(s)ds$.

We denote $J(\hat{u}) = \int_0^{t_1} \hat{u}^T(s)\hat{u}(s)ds = x_0^T L(t_1)x_0$.

(a) Show $W(0,t)$ satisfies $AW + WA^T + BB^T = e^{At}BB^Te^{A^Tt}$. ....(4p)

(b) Show that for any $x_0 \neq 0$, $x_0^T L(t_2)x_0 < x_0^T L(t_1)x_0$, $t_2 > t_1$. (Hint: for a nonsingular matrix $M(t)$, show that $\frac{d}{dt}(M^{-1}(t)) = -M^{-1}\dot{M}M^{-1}$.) ....(8p)

(c) It is obvious that $\lim_{t_1\to\infty} L(t_1) = 0$ if A is a stable matrix. What is $\lim_{t_1\to\infty} L(t_1)$ if $-A$ is a stable matrix? ....(8p)

Answer: See Exam of March 2012.

5. (a) Consider a controllable system

$$\dot{x} = Ax + Bu.$$

Show that for any $t_1 > 0$, $u = -B^TW^{-1}(t_1)x$ asymptotically stabilizes the system, where

$$W(t_1) = \int_0^{t_1} e^{-At}BB^Te^{-A^Tt}dt.$$

.... (6p)

Answer: See Exam of April 2015.

(b) Consider a controllable system

$$\dot{x} = Ax + Bu$$
$$y = Cx,$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, and $C \neq 0$. Show that we can always find a matrix K such that $(C, A + BK)$ is observable. .... (6p)

Answer: There must be one row in C that is not identically zero, assume $c_1 \neq 0$. Given any K_0 such that $A+BK_0$ has all distinguishable eigenvalues, for almost any $v \in R^m$, $(A+BK_0, Bv)$ is controllable. Then $\dot{x} = (A+BK_0)x + Bv\bar{u}$, $y = c_1x$ becomes a controllable SISO system. Then we can find $\bar{u} = k_1x$ such that no eigenvalue of $(A+BK+Bvk_1)$ coincides with the zeros of the SISO system. Then $(c_1, A+B(K_0+vk_1))$ is observable, thus $(C, A+B(K_0+vk_1))$ is observable.

(c) Let (A, B) be controllable. Assume that P_i, $i = 1, 2$ is the positive definition solution to the following ARE:

$$A^TP_i + P_iA - P_iBB^TP_i + Q_i = 0, \ i = 1, 2.$$

Show that if $Q_2 \geq Q_1$, then $P_2 \geq P_1$. ....(8p)

Answer: For any x_0, $x_0^TP_ix_0$ is the optimal cost for

$$\begin{aligned} min \quad & \int_0^\infty (x^TQ_ix + u^Tu)dt \\ st \quad & \dot{x} = Ax + Bu, \ x(0) = x_0 \end{aligned}$$

Since $\int_0^\infty (x^T Q_2 x + u^T u)dt = \int_0^\infty (x^T Q_1 x + u^T u)dt + \int_0^\infty x^T (Q_2 - Q_1) x dt \geq \int_0^\infty (x^T Q_1 x + u^T u)dt$, we have $min \ \int_0^\infty (x^T Q_2 x + u^T u)dt \geq min \ \int_0^\infty (x^T Q_1 x + u^T u)dt$. Thus $P_2 \geq P_1$.