KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
Part one: 08.00-10:00, April 15, 2020

Examiner: Xiaoming Hu, tel. 0707967831.

Important! This exam consists of two parts. The second part starts at 10:20 and you will receive questions for Part two after the exam on Part one is concluded. You write the solutions on paper and then upload the scanned (or photoed) solutions in pdf format to Canvas. You must upload the solutions to Part one before Part two of the exam starts! You should name the file as "lastname_firstname_x", where "x" is either 1 or 2.

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

---

1. Determine if each of the following statements is true or false. **You must motivate your answers. No motivation no point.**

(a) Consider an n-dimensional *time-varying* system $\dot{x}(t) = A(t)x(t)$, where $A(t)$ is continuous. If $\|x(t)\|^2 = \|x(t_0)\|^2 \ \forall t \in R$ and $\forall x(t_0) \in R^n$, then $det\ A(t) = 0 \ \forall t \in R$, where "det" means determinant. . . . . (5p)

Answer: False, for example, $n = 2$ and $a_{11} = a_{22} = 0, a_{12} = -a_{22} \neq 0$.

(b) Consider $\dot{x} = Ax + bu$, where $x \in R^n,\ u \in R$. If all eigenvalues of A are real and distinct (no identical eigenvalues), then there exists b such that (A, b) is controllable. . . . . (5p)

Answer: True. We can transform A into a diagonal matrix, and choose $\bar{b} = [1 \ \cdots \ 1]^T$.

(c) For a linear time-invariant system $\dot{x} = Ax + Bu,\ y = Cx$, where $x \in R^n,\ u \in R^m,\ y \in R^p$, both controllability and observability are invariant under feedback control, namely if (A, B) is controllable and (C, A) is observable then $(A + BK, B)$ is controllable and $(C, A + BK)$ is observable as well for any K. (5p)

Answer: False. Observability is not invariant under feedback control. For example, with the feedback control in 2(e), the observability is lost.

(d) Consider the optimal control problem for $\dot{x} = Ax + Bu,\ x(0) = x_0$:

$$\min_u \int_0^\infty (x^T Q x + u^T R u) dt,$$

where $Q \geq 0$ and $R > 0$. A necessary condition for the existence of optimal controller is that (A, B) is controllable. . . . . (5p)

Answer: False, for example, when A is a stable matrix.

2. Consider :

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned}$$

where $x \in R^3$, $u \in R$, $y \in R$, and the transfer function is

$$r(s) = c(sI - A)^{-1}b = \frac{s + \alpha}{s^3 + 3s^2 + 3s + 1},$$

where α is a constant. Assume $c = \begin{bmatrix} \alpha & 1 & 0 \end{bmatrix}$.

(a) Find matrices A and b. .... (5p)
Answer: A and b are in the canonical controllable form, where the last row of A is $[-1 \ -3 \ -3]$, and $b = [0 \ 0 \ 1]^T$.

(b) Show $\lim_{t\to\infty} e^{At}x_0 = 0$ for all $x_0 \in R^3$. .... (3p)
Answer: Since all the eigenvalues of A are -1, this can be shown easily.

(c) For what α is (c, A) observable? .... (3p)
Answer: $\alpha \neq 1$.

(d) Can we find $u(t) = Kx$ that makes $\mathcal{D} = \{x \in R^3 : cx = 0\}$ invariant and why? i.e., $cx(t) = 0, \forall t \geq 0$ if $cx(0) = 0$. .... (2p)
Answer: No, since we can find initial states in $\mathcal{D}$ such that $\dot{y}(t)|_{t=0} \neq 0$, which are independent of the choice of K.

(e) If your answer is no in (d), find $u(t) = Kx$ that makes $cx(t) = 0, \forall t \geq 0$ for as many initial states $x(0)$ in $\mathcal{D}$ as possible. .... (3p)
Answer: For example, $u = x_1 + 3x_2 + (3-\alpha)x_3$, s.t. $\alpha x_1 + x_2 = 0$, $\alpha x_2 + x_3 = 0$.

(f) Discuss for what α is every solution $x(t)$ obtained in (e) bounded (you must prove your conclusion). A solution $x(t)$ obtained in (e) means that $u(t) = Kx$ is used and $cx(t) = 0, \forall t \geq 0$. .... (4p)
Answer: $\alpha > 0$.