KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
Part two: 10.20-13:20, April 15, 2020

Examiner: Xiaoming Hu, tel. 0707967831.

Important! This exam consists of two parts. The second part starts at 10:20 and you will receive questions for Part two after the exam on Part one is concluded. You write the solutions on paper and then upload the scanned (or photoed) solutions in pdf format to Canvas. You must upload the solutions to Part one before Part two of the exam starts! You should name the file as "lastname_firstname_x", where "x" is either 1 or 2.

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Solution methods: All conclusions should be carefully motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 45 points credit (including your still valid bonus) to pass the exam. The other grade limits are listed on the course home page.

---

1. Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{s+2} & \frac{2\gamma}{s+2} & \frac{1}{s+2} \\ \frac{1}{(s+1)(s+2)} & \frac{2}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \end{bmatrix},$$

where γ is a constant.

(a) Find the standard reachable realization. . . . . (6p)

Answer: omitted.

(b) Compute the McMillan degree of $R(s)$. . . . . (6p)

Answer: $\delta(R(s)) = 3$ if $\gamma \neq 1$, otherwise $\delta(R(s)) = 2$.

(c) For the case $\gamma = 1$, find a minimal realization of $R(s)$ and verify your answer by computing $C(sI - A)^{-1}B$. . . . . (8p)

Answer: In this case we in principal only need to find a minimal realization for one of the columns in $R(s)$, then expand the B matrix to include all three inputs. A standard reachable realization of column one gives: $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $b_1 = (0\ 1)^T$.

2. Consider the optimal control problem

$$\min_u J = \int_0^\infty (\epsilon y^2 + u^2)dt$$
$$\text{s.t.}$$
$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$x(0) = x_0,$$

where, $\epsilon > 0$, and

$$A = \begin{bmatrix} a_1 & 0 \\ 1 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(a) Show for $a_2 \leq 0$, the associated algebraic Riccati equation (ARE) does not have a positive definite solution. ....(5p)

Answer: Let $P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$. In this case one can easily show that $p_2 = p_3 = 0$.

(b) Let $P(\epsilon)$ denote the symmetric solution to the ARE. Show that $\lim_{\epsilon \to 0} P(\epsilon)$ is positive definite if and only if $a_1 > 0$ and $a_2 > 0$. ....(5p)

Answer: In this case one can rewrite the ARE as

$$-P^{-1}A^T - AP^{-1} = -BB^T.$$

Since $(-A, B)$ is controllable, P^{-1} is positive definite iff $-A$ is a stable matrix.

(c) Show when $a_1 \cdot a_2 \neq 0$, $\lim_{\epsilon \to 0}(A - BB^T P(\epsilon))$ has eigenvalues $\{-|a_1|, -|a_2|\}$. (6p)

Answer: Case 1: $a_1 > 0$ and $a_2 > 0$, then from (b) $A - BB^T P = -P^{-1}A^T P$. Case 2: $a_1 < 0$ and $a_2 < 0$, then $\lim_{\epsilon \to 0} P(\epsilon) = 0$. Case 3: $a_1 \cdot a_2 < 0$, decompose and calculate $\lim_{\epsilon \to 0} P(\epsilon)$ according to cases 1 and 2.

(d) Now let $\epsilon = 0$ and $a_2 < 0$, find the optimal control $u = K^* x_1$ when $a_1 \neq 0$, namely express K^* as a function of a_1. What happens if $a_1 = 0$? .... (4p)

Answer: $u = 0$ if $a_1 < 0$ and $u = -2a_1 x_1$ if $a_1 > 0$. Optimal control does not exist if $a_1 = 0$.

3. (a) Consider a 2×2 matrix of continuous functions

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & -a_{11}(t) \end{bmatrix}.$$

Show that $det\ \Phi(t, t_0) = 1\ \forall t, t_0$, where $\Phi(t, t_0)$ is the state transition matrix, and "det" denotes determinant. ....(5p)

Answer: Let $\Phi(t, t_0) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$, then $det\ \Phi(t, t_0) = \phi_{11}\phi_{22} - \phi_{12}\phi_{21}$. $\frac{d}{dt}(det\ \Phi(t, t_0)) = 0$.

(b) Consider a single-input-single-output system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

and its corresponding transfer function

$$G(s) = C(sI - A)^{-1}B = \frac{s^m + b_{m-1}s^{m-1} + \ldots + b_0}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times 1}$, $C \in \mathbf{R}^{1\times n}$. The relative degree $r = n - m$ is the excess degree of the denominator compared to the numerator.

(1) Show that

$$CA^kB = 0, \quad k = 0, 1, \ldots, r-2 \quad \text{and} \quad CA^{r-1}B \neq 0$$

and $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix}$ has full row rank. .... (5p)

Answer: The first part can be shown using the fact that $CA^{i-1}B = R_i$, where R_i are the so-called Markov parameters of the transfer function. To show the second part, assume $\sum_{i=1}^{r} \alpha_i CA^{i-1} = 0$. By multiplying both sides with $B,\ AB, \cdots$ recursively, we can show all α_i must be zero.

(2) Let $u = Kx$. Then the collection of all initial states such that $y(t) \equiv 0$, denoted by $D_K = \{x_0 : Ce^{(A+BK)t}x_0 = 0\ \forall t \geq 0\}$, is a subspace in R^n. Show that there exists $K = K^*$ such that the dimension of D_{K^*} is m. (hint: a function $f(t) \equiv 0$ implies $f^{(k)}(t) \equiv 0$) .... (5p)

Answer: We can compute that $y^{(k)}(t) = CA^kx(t),\ k = 0, \cdots, r-1$ and $y^{(r)}(t) = CA^rx(t) + CA^{r-1}Bu$. Then $y(t) \equiv 0$ implies that $CA^kx(t) \equiv 0,\ k = 0, \cdots, r-1$ and $u = -(CA^{r-1}B)^{-1}CA^rx(t)$, which gives D_K dimension $n - r$.

(c) Suppose (A, B) is controllable and (C, A) is observable, and P is the positive definite solution to

$$A^TP + PA - PBB^TP + C^TC = 0.$$

Show $A - kBB^TP$ is a stable matrix for all $k \geq 1$. .... (5p)

Answer: From the Riccati equaltion we have

$$(A - kBB^TP)^TP + P(A - kBB^TP) = (1-2k)PBB^TP - C^TC.$$

This shows that $A - kBB^TP$ is at least critically stable when $k \geq 1$. We then apply the same argument in p. 77 of the compendium to show that $A - kBB^TP$ is a stable matrix.