KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
Part two: 16.20-19:20, January 8, 2021

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Important! This exam consists of two parts. The second part starts at 16:20 and you will receive questions for Part two after the exam on Part one is concluded. You write the solutions on paper and then upload the scanned (or photoed) solutions in pdf format to Canvas. You must upload the solutions to Part one before Part two of the exam starts!

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory (pdf or paper version), Per Enqvist, Exercises in Mathematical Systems Theory (pdf or paper version), your own class notes (digital or paper version), and β mathematics handbook.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 40 points credit to pass the exam. The other grade limits are listed on the course home page.

The sub-problems in each problem are listed by the ascending order of difficulty whenever it is possible.

Matrix notation: We use $A(t)$ to denote a time-varying matrix and A to denote a constant matrix.

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1. (20p) Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{s+1} & \frac{1}{s+1} \\ \frac{1}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix},$$

where γ is a real constant.

(*a*) Find the standard reachable realization. . . . . (6p)
Answer: Omitted.

(*b*) Compute the McMillan degree of $R(s)$. . . . . (6p)
Answer: $\delta(R) = 3$ if $\gamma \neq 1$, otherwise $\delta(R) = 2$.

(*c*) For the case $\gamma = 1$, find a minimal realization of $R(s)$ and verify your answer by computing $C(sI - A)^{-1}B$. . . . . (8p)
Answer: Note that in this case $y_2 = y_1 - y_3$, so we only need to find a minimal realization for

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+2} & \frac{1}{s+2} \end{bmatrix}.$$

In turn we only need to find a minimal realization for

$$\begin{bmatrix} \frac{1}{s+1} \\ \\ \frac{1}{s+2} \end{bmatrix},$$

and the standard reachable realization will do. The rest is omitted.

2. (20p) Consider the optimal control problem

$$\min_u J = \int_0^{t_1} u^T u dt + x(t_1)^T S x(t_1) \quad \text{s.t.} \quad \dot{x} = Ax + Bu, \ x(0) = x_0,$$

where, $t_1 > 0$, and

$$A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad S = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$$

where a_1, a_2 are real constants and s_1, s_2 are non-negative constants. Let $u = -B^T P(t_1 - t)x$ denote the optimal control.

(*a*) Solve the corresponding dynamical Riccati equation to obtain $P(t_1 - t)$. (Note that for a general linear system $\dot{x} = Ax + Bu$, if $x(t_1)$ is given instead of $x(0)$, we just need to express $x(0)$ in terms of $x(t_1)$ by solving $x(t_1) = e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-s)}Bu(s)ds$) .... (8p)

Answer: From the adjoint system we can easily obtain $Y == exp(A^T(t_1-t))S$, then by plugging in $Y(t)$ into the equation for X and using the hint we have

$$X = exp(-A(t_1 - t)) + \int_0^{t_1-t} exp(-As)BB^T exp(-A^T s)ds \ exp(A^T(t_1 - t))S,$$

then plug in A and B. The rest is omitted.

(*b*) Discuss conditions on s_1, s_2 such that $P(t_1-t)$ is positive definite for $0 \le t < t_1$. (4p)

Answer: s_1, s_2 are positive.

(*c*) Compute $\lim_{t_1-t\to\infty} P(t_1-t)$ for the case A is a stable matrix and explain what the result implies in terms of the optimal control problem. .... (4p)

Answer: Since $X(t_1 - t)$ tends to ∞ and $Y(t_1 - t)$ tends to 0 in this case, $P(t_1 - t) = YX^{-1}$ tends to 0, which implies that the optimal control is $u = 0$ (do nothing), since $x(t)$ will anyway converge to 0.

(*d*) Assume s_1, s_2 are positive. Compute the eigenvalues of $\lim_{t_1-t\to\infty}(A-BB^T P(t_1-t))$ for the case $-A$ is a stable matrix and $a_1 \neq a_2$. What will happen if we take away the assumption $a_1 \neq a_2$? .... (4p)

Answer: In this case $P^{-1}(t_1 - t) = (YX^{-1})^{-1} = XY^{-1}$ tends to

$$\int_0^{\infty} exp(-As)BB^T exp(-A^T s)ds$$

which is the solution to $-P^{-1}A^T - AP^{-1} + BB^T = 0$. Thus, $A - BB^T P = -P^{-1}A^T P$, which has same eigenvalues as $-A^T$ thus as $-A$. If $a_1 = a_2$, then the system is not reachable thus the reachability Gramian will not be invertible.

3. (15p)

(a) Consider $\dot{x} = Ax,\ y = Cx$, where $x \in R^n$. We know that (C, A) is observable if and only if $ker\ \Omega = 0$. Show that $ker\ \Omega = 0$ only if $rank\ H(s) = \begin{pmatrix} C \\ sI - A \end{pmatrix} = n$ for all complex numbers s (here we only show the necessity despite that the condition is also sufficient). *Hint*: H obviously has full rank if s is not an eigenvalue of A, and what does it imply if for some eigenvalue s_0 H does not have full rank? .... (7p)

Answer: Assume that (C, A) is observable and we prove the necessity by contradiction. Suppose $rank\ H(s_0) < n$, then $\exists x_0 \neq 0$ such that $H(s_0)x_0 = 0$, namely $Cx_0 = 0$ and $Ax_0 = s_0x_0$. Multiply both sides of the second equality first by A, then A^2,..., we have $A^k x_0 = s_0^k x_0$. Then $CA^k x_0 = s_0^k Cx_0 = 0$, thus $x_0 \in ker\ \Omega$, contradiction.

(b) Consider the algebraic Riccati equation

$$A^T P + PA - PBR^{-1}B^T P + C^T C = 0.$$

Assume P is a real positive **semidefinite** solution and (C, A) is observable.

(1) Show that every positive semidefinite solution P is positive definite, i.e., $ker\ P = 0$. .... (5p)
Answer: Using techniques from an earlier exam we can show that $ker\ P \subset ker\ C$ and it is A-invariant. Thus $ker\ P \subset ker\ \Omega$, which implies $ker\ P = 0$.

(2) Show that every positive semidefinite solution P is a stabilizing solution, i.e., $A - BR^{-1}B^T P$ is a stable matrix. .... (3p)
Answer: From (1) we know every positive semidefinite solution P is also positive definite. Then we can use the technique in the compendium to prove the conclusion.