KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
Part two: 10:20-13:20, April 7, 2021

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Important! This exam consists of two parts. The second part starts at 10:20 and you will receive questions for Part two after the exam on Part one is concluded. You write the solutions on paper and then upload the scanned (or photoed) solutions in pdf format to Canvas. You must upload the solutions to Part one before Part two of the exam starts!

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory (pdf or paper version), Per Enqvist, Exercises in Mathematical Systems Theory (pdf or paper version), your own class notes (digital or paper version), and β mathematics handbook.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 40 points credit to pass the exam. The other grade limits are listed on the course home page.

Matrix notation: We use $A(t)$ to denote a time-varying matrix and A to denote a constant matrix.

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1. (20p) Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+\alpha} & \frac{1}{s+\alpha} \end{bmatrix},$$

where γ, α are real nonzero constants.

(a) Find the standard reachable realization for all possible values of γ, α. ....(7p)
Answer: We need to do this in two cases: 1. $\alpha = 1$, $\chi(s) = s + 1$, 2. $\alpha \neq 1$, $\chi(s) = s^2 + (\alpha + 1)s + \alpha$. The rest is omitted.

(b) Compute the McMillan degree of $R(s)$ for all possible values of γ, α. .....(6p)
Answer: When $\gamma = 1$ and $\alpha = 1$, $\delta(R) = 1$, otherwise $\delta(R) = 2$.

(c) Let $\alpha = 1$. For the case the realization in (a) is not observable, find a minimal realization of $R(s)$. ....(7p)
Answer: When $\alpha = 1$, the realization in (a) is two dimensional. It will be unobservable if $\gamma = 1$ (since $\delta(R) = 1$). It is straight forward to derive the one-dimensional realization where $A = -1$.

2. (20p) Consider the optimal control problem

$$\min_u J = \int_0^{t_1} u^2 dt + x_1(t_1)^2 + x_2(t_1)^2 \quad \text{s.t.} \quad \dot{x} = Ax + Bu,\ x(0) = x_0,$$

where, $t_1 > 0$, and

$$A = \begin{bmatrix} \alpha & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where α is real constant. Let $u = -B^T P(t_1 - t)x$ denote the optimal control.

(*a*) For the case $\alpha = 0$, find $P(t_1 - t)$ that is the solution to the corresponding dynamical Riccati equation. (Note that for a general linear system $\dot{x} = Ax + Bu$, if $x(t_1)$ is given instead of $x(0)$, we just need to express $x(0)$ in terms of $x(t_1)$ by solving $x(t_1) = e^{At_1}x(0) + \int_0^{t_1} e^{A(t_1-s)}Bu(s)ds$) ....(6p)

Answer: We use the ajoint system to solve for $P(t_1 - t)$. We compute first that $exp(At) = (\cos(t)\ \sin(t); -\sin(t)\ \cos(t))$ and note that $exp(A^T t) = exp(-At)$. We obtain first $Y = exp(A^T(t_1 - t))$ $(Y(t_1) = I)$, then by plugging in $Y(t)$ into the equation for X and using the hint we have

$$X = exp(-A(t_1 - t)) + \int_0^{t_1 - t} exp(-As)BB^T exp(-A^T s)ds\ exp(A^T(t_1 - t))S,$$

then plug in A and B. The rest is omitted.

(*b*) For the case $\alpha = 0$, does $\lim_{t_1 - t \to \infty} P(t_1 - t)$ exist? .... (3p)

Answer: No, since trigonometrical functions do not have limit as t tends to infinity.

(*c*) Show that $\lim_{t_1 - t \to \infty} P(t_1 - t)$ exits if $\alpha \neq 0$ (you do not have to compute $P(t_1 - t)$ explicitly here). .... (6p)

Answer: 1. When $\alpha < 0$, two eigenvalues of A have negative real parts, then $Y \to 0$ and $X \to \infty$, thus $P = YX^{-1} \to 0$; 2. When $\alpha > 0$, two eigenvalues of A have positive real parts, then $exp(-A(t_1 - t)) \to 0$, thus $P^{-1} = XY^{-1} = \int_0^\infty exp(-As)BB^T exp(-A^T s)ds$, which is the solution to $-P^{-1}A^T - AP^{-1} + BB^T = 0$.

(*d*) For the case $\alpha > 0$, compute the eigenvalues of $\lim_{t_1 - t \to \infty}(A - BB^T P(t_1 - t))$. (5p)

Answer: From the last equation in (c) we have $A - BB^T P = -P^{-1}A^T P$, thus it has the same eigenvalues as $-A$.

3. (15p)

(*a*) For any $a = (a_1\ a_2\ a_3)^T$, we can generate a skew symmetric matrix as follows

$$A = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

Now suppose $\|a\| = 1$. Show that

$$e^{At} = I + A\sin(t) + A^2(1 - \cos(t)).$$

(Hint: What is A^3? Taylor expansion for $\sin(t)$ and $\cos(t)$ can be useful). (6p)

Answer: Let $e^{At} = I + At + \cdots + A^k \frac{t^k}{k!} + \cdots$. We can compute that $A^3 = -A$, then $A^4 = A(-A) = -A^2$, which leads to $A^{2k+2} = (-1)^k A^2$ $(k > 0)$, and $A^{2k+1} = (-1)^k A$. Then $e^{At} = I + A(t - \frac{t^3}{3!} + \cdots + (-1)^k \frac{t^{2k+1}}{(2k+1)!} + \cdots) + A^2(\frac{t^2}{2} + \cdots + (-1)^k \frac{t^{2k+2}}{(2k+2)!} + \cdots) = I + A\sin(t) + A^2(1 - \cos(t))$

(*b*) Consider a one-dimensional system

$$\begin{aligned} x(t+1) &= ax(t) \\ y(t) &= x(t) + w(t), \end{aligned}$$

where $|a| > 1$, both $x(0)$ and $w(t)$ are Gaussian with zero mean and covariances p_0 and σ respectively. respectively.

(*i*) Write down the Kalman filter $\hat{x}(t)$ for $x(t)$. . . . . (2p)
Answer: This is to write down (70), (71) and (74) in the compendium.

(*ii*) Express the covariance matrix $p(t) = E\{(x(t) - \hat{x}(t))^2\}$ in terms of t, a, p_0, σ. (4p)
Answer: $p^{-1}(t+1) = a^{-2}(p^{-1}(t) + \sigma^{-1})$. Then, $\frac{1}{p(t)} = a^{-2t} p_0^{-1} + \sigma^{-1} \sum_{i=1}^{t} a^{-2i}$.

(*iii*) Show that $\lim_{t\to\infty} |a - ak(t)|$ exists and is strictly less than 1 (where $k(t)$ is the Kalman gain). . . . . (3p)
Answer: Let $p^{-1} = a^{-2}(p^{-1} + \sigma^{-1})$ or $p = \frac{\sigma a^2 p}{p + \sigma}$ be the steady state $p(t)$, then $p = \sigma(a^2 - 1)$, thus $\lim_{t\to\infty} |a - ak(t)| = \frac{1}{|a|} < 1$.