KTH Matematik

Solution to Exam in SF2832 Mathematical Systems Theory
14.00-19:00, January 10, 2023

Examiner: Xiaoming Hu, tel. 0707967831.

Allowed material: Anders Lindquist & Janne Sand, An Introduction to Mathematical Systems Theory, Per Enqvist, Exercises in Mathematical Systems Theory, your own class notes, and β mathematics handbook.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

You need 40 points credit to pass the exam. The other grade limits are listed on the course home page.
The sub-problems in each problem are listed by the ascending order of difficulty whenever it is possible.

Matrix notation: We use $A(t)$ to denote a time-varying matrix and A to denote a constant matrix.

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1. (20p) Determine if each of the following statements is true or false. **You must motivate your answers. No motivation no point.**

(a) Consider $\dot{x} = Ax$ where $x \in R^n$. If $x = 0$ is unstable, then A must have at least one eigenvalue with positive real part. . . . . (6p)
Answer: False, for $A = (0\ 1; 0\ 0)$.

(b) Consider

$$\dot{x} = Ax$$
$$y = Cx,$$

where $x \in R^n,\ y \in R^p$. Assume $x(0) = x_0 \notin ker\ \Omega$, then $y(t) = Ce^{At}x_0$ can not be identically zero over any time interval $[t_1, t_2]$, where $t_2 > t_1 \geq 0$. . . (4p)
Answer: True, since $Ce^{At}x_0 = 0\ \forall t \in [t_1, t_2]$ iff $x_0 \in ker\ \Omega$.

(c) Consider $\dot{x} = Ax + Bu,\ y = Cx$, where $x \in R^n,\ u \in R,\ y \in R$.
c1. Assume that (C, A) is observable, then $rank\ A \geq n - 1$. . . . . (3p)
Answer: True, since (C, A) is observable means (A^T, C^T) is controllable and $rank\ A^T = rank\ A$.
c2. Let $C(sI - A)^{-1}B = \frac{n(s)}{d(s)}$, where $d(s) = det\ (sI - A)$. If $n(s) = 1$, then $(C, A + BK)$ is observable for any K. . . . . (3p)
Answer: True. There are several ways to see this. With $n(s) = 1$ we know that the McMillan degree is n, thus the standard controllable realization is minimal (remember that all minimal realizations are similar). Since $C = (1\ 0\ \cdots\ 0)$, $(C, A + BK)$ is observable for any K.

(*d*) Consider the optimal control problem for $\dot{x} = Ax + Bu,\ x(t_0) = x_0$:

$$\min_u \int_{t_0}^{t_1} (x^T C^T Cx + u^T u)dt,$$

where $t_0 < t_1 < \infty$, and (C, A) is observable. Let $x_0^T P(t_0)x_0$ be the optimal cost. If (A, B) is not controllable, then $P(t_0)$ is not positive definite. .....(4p)
Answer: False. In the finite time interval case, it is observability that determines if $P(t_0)$ is positive definite.

2. (25p) Consider :

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned} \qquad (1)$$

where $x \in R^4,\ u \in R,\ y \in R$, and the transfer function is

$$r(s) = c(sI - A)^{-1}b = \frac{s^2 + \alpha s + 1}{(s^2 + 3s + 2)(s^2 + 1)},$$

where α is a real constant. Assume $b = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T$.

(*a*) Find matrices A and c. .... (5p)
Answer: $c = (1\ \alpha\ 1\ 0)$, last row of A is $(-2\ -3\ -3\ -3)$.

(*b*) For what α is your (c, A) observable? .... (6p)
Answer: $\alpha \neq 0, 2, 2.5$.

(*c*) Verify if it is true that $\|e^{At}x_0\| \leq M\|x_0\|$ for all $x_0 \in R^4$ and all $t \geq 0$, where M is a positive constant. .... (4p)
Answer: It is true since A has eigenvalues $\pm j, -1, -2$.

(*d*) Find $u(t) = Kx$ that makes $\mathcal{D} = \{x \in R^4 : cx = 0\}$ invariant, i.e., $cx(t) = 0, \forall t \geq 0$ if $cx(0) = 0$. .... (3p)
Answer: Apparently no u can make the entire $\mathcal{D}$ invariant (4p). We need to obtain u from $\ddot{y} = 0$, i.e., $x_3 + \alpha x_4 - 2x_1 - 3x_2 - 3x_3 - 3x_4 + u = 0$, which is a subspace in $\mathcal{D}$.

(*e*) Are the state trajectories $x(t)$ obtained in (d) bounded (you must prove your conclusion)? A solution $x(t)$ obtained in (d) means a solution to (1) when $u(t) = Kx$ is used and $cx(0) = 0$. .... (4p)
Answer: For the trajectories staying in $\mathcal{D}$, yes if $\alpha \geq 0$.

(*f*) For what α can we find a three dimensional minimal realization of the $r(s)$ such that for the minimal realization there exists $u(t) = Kx$ that makes both $cx(t) = 0, \forall t \geq 0$ and $\lim_{t\to\infty} x(t) = 0$? .... (3p)
Answer: $\alpha = 2, 2.5$.

3. (20p) Consider the transfer matrix

$$R(s) = \begin{bmatrix} \frac{\gamma}{s+2} & \frac{1}{(s+1)(s+2)} \\ \frac{\gamma+1}{s+2} & \frac{2}{(s+1)(s+2)} \\ \frac{1}{s+2} & \frac{1}{(s+1)(s+2)} \end{bmatrix},$$

where γ is a real constant.

(*a*) Find the standard reachable realization. .... (6p)

Answer: omitted.

(*b*) Compute the McMillan degree of $R(s)$. .... (6p)

Answer: $\delta(R) = 2$ if $\gamma = 1$ otherwise $\delta(R) = 3$.

(*c*) For the case $\gamma = 1$, find a minimal realization of $R(s)$ and verify your answer by computing $C(sI - A)^{-1}B$. (Hint: one does not have to use Kalman decomposition) .... (8p)

Answer: In this case $\delta(R) = 2$ and all three rows are essentially the same. So we just need to find a minimal realization for $(\frac{1}{s+2} \;\; \frac{1}{(s+1)(s+2)})$, namely a realization of dimension 2. The standard observable realization would do it. The rest is omitted.

4. (20p) Consider the optimal control problem

$$\min_u J = \int_0^{t_1} u^T u dt + x(t_1)^T S x(t_1) \quad \text{s.t.} \quad \dot{x} = Ax + Bu,\ x(0) = x_0,$$

where (A, B) is controllable and S is positive definite.

Let $u = -B^T P(t_1 - t)x$ denote the optimal control.

(*a*) Solve the Riccati equation to obtain $P(t_1 - t)$. (**Hint:** to determine P is the same as determining P^{-1} if P is invertible) .... (10p)

Answer: By using the adjoint system, we have $Y = exp(A^T(t_1 - t))S$, $X = exp(-A(t_1 - t)) + \int_0^{t_1-t} exp(-As)BB^T exp(-A^T s)ds exp(A^T(t_1 - t))S$.

$P^{-1} = XY^{-1} = exp(-A(t_1-t))S^{-1}exp(-A^T(t_1-t)) + \int_0^{t_1-t} exp(-As)BB^T exp(-A^T s)ds$.

(*b*) Compute $\lim_{t_1-t\to\infty} P(t_1 - t)$ for the case A is a stable matrix. .... (4p)

Answer: If A is stable, $P^{-1} \to \infty$.

(*c*) What are the eigenvalues of $\lim_{t_1-t\to\infty}(A - BB^T P(t_1 - t))$ for the case $-A$ is a stable matrix? .... (6p)

Answer: If $-A$ is stable, $P^{-1} \to \int_0^\infty exp(-As)BB^T exp(-A^T s)ds$, which satisfies $-P^{-1}A^T - AP^{-1} + BB^T = 0$. Thus, $A - BB^T P = -P^{-1}A^T P$, which has same eigenvalues as $-A^T$ thus as $-A$.

5. (15p)

(*a*) Let $B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$ and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where B_1 is an $n_1 \times m$ matrix with rank n_1. The matrices A_{21} and A_{22} have dimensions $(n-n_1)\times n_1$ and $(n-n_1)\times(n-n_1)$ respectively. Show that (A, B) is reachable if and only if (A_{22}, A_{21}) is reachable. (5p)

Answer: We can find $u = Fx$ such that $A + BF = \begin{bmatrix} 0 & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. There are several ways to show $(A+BF, B)$ is reachable iff (A_{22}, A_{21}) is reachable. Here we show $(B^T, (A+BF)^T)$ is observable iff (A_{21}^T, A_{22}^T) is observable: $y = B_1^T x_1 = 0$ implies $x_1 = 0$. Thus $A_{21}^T x_2 = 0$, while $\dot{x}_2 = A_{22}^T x_2$. In this case $x_2 = 0$ iff (A_{21}^T, A_{22}^T) is observable.

(*b*) Let (A, B) be controllable, $Q_i \geq 0$ and (Q_i, A) be observable for $i = 1, 2$. Assume that $P_i,\ i = 1, 2$ is respectively the positive definition solution to the following ARE:

$$A^T P_i + P_i A - P_i B B^T P_i + Q_i = 0,\ i = 1, 2.$$

Show that if $Q_2 \geq Q_1$ $(Q_2 - Q_1 \geq 0)$, then $P_2 \geq P_1$. .... (4p)
Answer: For any x_0, $x_0^T P_i x_0$ is the optimal cost for

$$\begin{aligned} min \quad & \int_0^\infty (x^T Q_i x + u^T u) dt \\ st \quad & \dot{x} = Ax + Bu,\ x(0) = x_0 \end{aligned}$$

Since $\int_0^\infty (x^T Q_2 x + u^T u)dt = \int_0^\infty (x^T Q_1 x + u^T u)dt + \int_0^\infty x^T (Q_2 - Q_1) x dt \geq \int_0^\infty (x^T Q_1 x + u^T u)dt$, we have $min\ \int_0^\infty (x^T Q_2 x + u^T u)dt \geq min\ \int_0^\infty (x^T Q_1 x + u^T u)dt$. Thus $P_2 \geq P_1$.

(*c*) Consider a one-dimensional system

$$\begin{aligned} x(t+1) &= ax(t) \\ y(t) &= x(t) + w(t), \end{aligned}$$

where $a \neq 1$ and is non-zero, both $x(0)$ and $w(t)$ are Gaussian with zero mean and covariances p_0 and σ respectively.

(*i*) Write down the Kalman filter $\hat{x}(t)$ for $x(t)$. .... (1p)
Answer: Omitted.

(*ii*) Express the covariance matrix $p(t) = E\{(x(t) - \hat{x}(t))^2\}$ in terms of t, a, p_0, σ. (3p)
Answer: We have $p(t+1) = \frac{a^2 \sigma p(t)}{\sigma + p(t)}$, or $p(t+1)^{-1} = a^{-2} p(t)^{-1} + \frac{1}{a^2 \sigma}$, which is a linear equation. Thus $p(t)^{-1} = a^{-2t} p_0^{-1} + \sum_{i=0}^{t-1} a^{-2(t-1-i)} \frac{1}{a^2 \sigma}$.

(*iii*) Show that $\lim_{t\to\infty} |a - ak(t)| < 1$ (where $k(t)$ is the Kalman gain). . . (2p)
Answer: As $t \to \infty$ we have $p = \frac{a^2 \sigma}{\sigma + p}$. Thus $p = 0$ if $|a| < 1$ and $p = (a^2 - 1)\sigma$ if $|a| > 1$. Since $k = \frac{p}{p + \sigma}$, we only need to compute the case $|a| > 1$. We have $\lim_{t\to\infty} |a - ak(t)| = |\frac{1}{a}|$.

Good luck!