CHAPTER 2

# Invariant and controlled invariant subspaces

In this chapter we introduce two important concepts: invariant subspace and controlled invariant subspace, which will be used later on to solve many control problems.

## 2.1. Invariant subspaces

Consider an $n$-dimensional linear system

$$\dot{x} = Ax \tag{2.1}$$

where $x \in \mathbb{R}^n$.

DEFINITION 2.1. *A set $\Omega \subseteq \mathbb{R}^n$ is called an* invariant set *of* (2.1) *if for any initial condition $x_0 \in \Omega$, we have $x(x_0, t) = e^{At}x_0 \in \Omega,\ \forall t \geq 0$.*

Some trivial examples of invariant sets are $\mathbb{R}^n$ and $x = \{0\}$.

In this course we only consider a special class of invariant sets: invariant subspaces. Now let us discuss conditions for a subspace $\mathcal{S}$ to be invariant. Since by Taylor expansion we have

$$x(x_0, t) = x_0 + tAx_0 + \frac{t^2}{2}A^2x_0 + \cdots,$$

it is obvious that if $A^i x_0 \in S\ \forall i \geq 0$, then $x(x_0, t) \in S,\ \forall t \geq 0$. Naturally this argument is true only if $\mathcal{S}$ is a linear subspace. It is easy to see as a sufficient condition

$$Az \in \mathcal{S}\ \forall z \in \mathcal{S}. \tag{2.2}$$

In other words, this condition implies that if we define a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$: $w = Az$, then the image of $S \subseteq \mathbb{R}^n$ is contained in $\mathcal{S}$. We denote this by

$$A\mathcal{S} \subseteq \mathcal{S}. \tag{2.3}$$

Now we show this condition is also necessary for $S$ to be invariant.

PROPOSITION 2.1. *A necessary and sufficient condition for a linear subspace $\mathcal{S}$ to be invariant under* (2.1) *is that condition* (2.3) *holds.*

PROOF

We only show the necessity here. Suppose there exists a point $x_0 \in \mathcal{S}$ such that $Ax_0 \notin \mathcal{S}$. Then when $t$ is sufficiently small, we have

$$x(x_0, t) = x_0 + tAx_0 + \mathcal{O}(t^2),$$

which does not belong to $\mathcal{S}$, since $\mathcal{S}$ is closed. ∎

EXAMPLE 2.1. *Consider*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{\cdot} = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

*Show that* $S = span\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ *is invariant.*

We first use the definition of invariant set to show this. It is easy to see that the set can be redefined as $S = \{x \in \mathbb{R}^2 : x_1 + x_2 = 0\}$. Then to show $S$ to be invariant is to show $x_1(x_0, t) + x_2(x_0, t) = 0 \ \forall t \geq 0$ if $x_0 \in S$. This is equivalent to showing $\dot{x}_1 + \dot{x}_2 = 0$ for all $(x_1, x_2)^T \in S$. We have $\dot{x}_1 + \dot{x}_2 = -2x_2 - x_2 + x_1 = -(x_1 + x_2) = 0$ if $(x_1, x_2)^T \in S$.

We can also show this with Proposition 2.1, since $A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = - \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

REMARK 2.1. *As an example to show the above result is only true for subspaces, we consider a circle defined by* $R = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$. *It is easy to show (as an exercise) that this set is invariant under the system*

$$\begin{aligned} \dot{x}_1 &= \omega x_2 \\ \dot{x}_2 &= -\omega x_1, \end{aligned}$$

*where* $\omega$ *is any positive number.*

*We ask the reader to check if* $Az \subset R$ *for any* $z \in R$.

Then we can use condition (2.3) as an alternative definition for invariant subspace.

DEFINITION 2.2. *A linear subspace* $\mathcal{S}$ *is* $A$-invariant *(invariant under* $\dot{x} = Ax$*) iff* $A\mathcal{S} \subseteq \mathcal{S}$.

## 2.2. Controlled invariant subspaces

Now we consider a control system

$$\dot{x} = Ax + Bu \tag{2.4}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

DEFINITION 2.3. $\mathcal{S}$ *is called a* controlled invariant subspace *of* (2.4) *if there exists a feedback control* $u = Fx$ *such that* $\mathcal{S}$ *is an invariant subspace of*

$$\dot{x} = (A + BF)x.$$

Similar to invariant subspace, we can also give another equivalent definition.

DEFINITION 2.4. $\mathcal{S}$ *is an* $(A, B)$*-invariant (controlled invariant) subspace if there exists a matrix* $F$ *such that*

$$(A + BF)\mathcal{S} \subseteq \mathcal{S}. \tag{2.5}$$

*Such an* $F$ *is called a* **friend** *of* $\mathcal{S}$.

We denote the set of friends by $\mathcal{F}(\mathcal{S})$. The following theorem provides a fundamental characterization of $(A, B)$-invariant subspaces that removes the explicit involvement of the feedback matrix $F$.

THEOREM 2.2. *$\mathcal{S}$ is $(A, B)$-invariant if and only if*

$$A\mathcal{S} \subseteq \mathcal{S} + \text{Im } B. \tag{2.6}$$

PROOF
Necessity: Suppose $F$ is a friend, then

$$(A + BF)\mathcal{S} \subseteq \mathcal{S}.$$

or

$$A\mathcal{S} \subseteq \mathcal{S} - B(F\mathcal{S}).$$

Since $B(F\mathcal{S}) \subseteq \text{Im } B$, thus (2.6) holds.
Sufficiency: The proof is constructive and is given as follows. ∎

We now give an algorithm for finding a friend of $\mathcal{V}$, which also serves as a proof of the sufficiency of Theorem 2.2.

*Algorithm for finding $F$*
Let $\{v_1, v_2, \dots, v_r\}$ be a basis for $\mathcal{V}$. Since $\mathcal{V}$ satisfies $A\mathcal{V} \subseteq \mathcal{V} + \text{Im } B$, there is for each $i = 1, \dots, r$ a $w_i \in \mathcal{V}$ and a $u_i \in \mathbb{R}^m$ such that

$$Av_i = w_i + Bu_i.$$

Let $F$ be a $m \times n$-matrix such that $Fv_i = -u_i$ for $i = 1, 2, \dots, r$ (if $r < n$ then $F$ is not unique). Then $Av_i = w_i - BFv_i$, i.e., $(A + BF)v_i = w_i \in \mathcal{V}$ and therefore $(A + BF)\mathcal{V} \subseteq \mathcal{V}$.

EXAMPLE 2.2. *Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Is the subspace $\mathcal{V} = \{x \in \mathbb{R}^2 : x_1 = x_2\}$ $(A, B)$-invariant? If so, find a friend of $\mathcal{V}$.*

*Clearly, $\mathcal{V}$ is spanned by $v = \begin{bmatrix} 1 & 1 \end{bmatrix}'$. Since*

$$Av = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathcal{V} + \text{Im } B,$$

*$\mathcal{V}$ is $(A, B)$-invariant, and we can let $u = 2$. To find $F$ we must solve the under-determined system of equations $Fv = -u$, i.e.,*

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = -2.$$

*The set $\mathcal{F}(\mathcal{V})$ is the affine space $\{(\lambda - 2, -\lambda); \lambda \in \mathbb{R}\}$. Choose, e.g., $F = \begin{bmatrix} -2 & 0 \end{bmatrix}$.*
*Is the subspace $\mathcal{V} := \{x \in \mathbb{R}^2 : x_1 = 0\}$ $(A, B)$-invariant?*

### 2.3. Reachability subspaces

In the rest of this chapter, we study the most elementary class of controlled invariant subspaces: *reachability (controllability) subspace.*

DEFINITION 2.5. *We use the notation $\langle A|S\rangle$ to denote the minimal $A$-invariant subspace that contains subspace $S$.*

Naturally if $S$ is already $A$-invariant, then $\langle A|S\rangle = S$. $\langle A|S\rangle$ can be computed in the following way:

(1) Let $S_0 = S$, check if $AS_0 \subseteq S_0$. If yes, stop. Otherwise,
(2) Let $S_{k+1} = AS_k + S_k$, $k \geq 0$.
(3) Check if $AS_{k+1} \subseteq S_{k+1}$. If yes, stop. Otherwise return to step 2.

Consider again (2.4). Recall that the reachable (controllable) subspace of (2.4) can be defined with our notation as

$$\langle A|\,\mathrm{Im}\ B\rangle = span\{B, AB, \cdots, A^{n-1}B\},$$

namely, the minimal $A$-invariant subspace that contains Im $B$. However, for many complex control problems, such as the problem of controllability under constraints discussed in the introduction, more refined study of reachability is needed.

Now consider the feedback law

$$u = Fx + Gv. \tag{2.7}$$

The corresponding closed-loop system

$$\dot{x} = (A + BF)x + BGv$$

has the reachable subspace

$$\mathcal{R} = \langle A + BF|\,\mathrm{Im}\ BG\rangle. \tag{2.8}$$

REMARK 2.2. *By construction, $\mathcal{R}$ is $(A, B)$-invariant.*

DEFINITION 2.6. *A subspace $\mathcal{R}$ is called a* reachability subspace *of (2.4) if there are $F$ and $G$ such that* (2.8) *holds.*

EXAMPLE 2.3. *If $G = I$ then*

$$\mathcal{R} = \langle A + BF|\,\mathrm{Im}\ B\rangle = \langle A|\,\mathrm{Im}\ B\rangle,$$

*is the reachable subspace. If $G = 0$ then $\mathcal{R} = 0$. For a* SISO*-system it is obvious that these are the only possible reachability subspaces.*

We now proceed with the analysis of reachability subspaces. The first theorem shows that the matrix $G$ can be removed from the characterization of $\mathcal{R}$ at the price of an implicit characterization, which however is of great use.

THEOREM 2.3. *A subspace $\mathcal{R}$ is a reachability subspace if and only if there is an $F$ such that*

$$\mathcal{R} = \langle A + BF|\,\mathrm{Im}\ B \cap \mathcal{R}\rangle. \tag{2.9}$$

PROOF
Necessity: Suppose $\mathcal{R}$ is a reachability subspace, i.e.,

$$\mathcal{R} = \langle A + BF | \operatorname{Im} BG \rangle \tag{2.10}$$

for some $F$ and $G$. Then $\operatorname{Im} BG \subseteq \mathcal{R}$ and $\operatorname{Im} BG \subseteq \operatorname{Im} B$, i.e.,

$$\operatorname{Im} BG \subseteq \operatorname{Im} B \cap \mathcal{R}.$$

Hence,

$$\mathcal{R} \subseteq \langle A + BF | \operatorname{Im} B \cap \mathcal{R} \rangle. \tag{2.11}$$

But $\mathcal{R}$ is $(A, B)$-invariant and therefore

$$(A + BF)^k \mathcal{R} \subseteq \mathcal{R} \text{ for } k \geq 1$$

and

$$\langle A + BF | \operatorname{Im} B \cap \mathcal{R} \rangle \subseteq \mathcal{R}. \tag{2.12}$$

Now (2.9) follows from (2.11) and (2.12).

Sufficiency: Suppose that (2.9) holds. It is enough to show that there is a $G$ such that $\operatorname{Im} B \cap \mathcal{R} = \operatorname{Im} BG$, since this will imply (2.10).

Let $p_1, p_2, \ldots, p_q$ be a basis for $\operatorname{Im} B \cap \mathcal{R}$. Then there is a linearly independent set $\{u_1, u_2, \ldots, u_q\}$ such that

$$p_i = Bu_i \qquad i = 1, 2, \ldots, q,$$

since if the $u_i$'s were linearly dependent then the $p_i$'s would be linearly dependent as well. If we let the input space be $R^m$ it holds that $q \leq \dim(\operatorname{Im} B) \leq m$. Choose $u_{q+1}, \ldots, u_m$ such that $\{u_1, \ldots, u_m\}$ is a basis for $\mathbb{R}^m$. We want

$$BGu_i = \begin{cases} p_i & i = 1, 2, \ldots, q, \\ 0 & i = q + 1, \ldots, m \end{cases}$$

which yields $\operatorname{Im} BG = \operatorname{Im} B \cap \mathcal{R}$, i.e.,

$$BG[u_1, \ldots, u_m] = [p_1, \ldots, p_q, 0, \ldots, 0] = B[u_1, \ldots, u_q, 0, \ldots, 0].$$

This is achieved by

$$G := [u_1, \ldots, u_q, 0, \ldots, 0][u_1, u_2, \ldots, u_m]^{-1}.$$

∎

We know that the reachability subspace $\mathcal{R}$ is $(A, B)$-invariant and it is obvious that $F$ in

$$\mathcal{R} = \langle A + BF | \operatorname{Im} B \cap \mathcal{R} \rangle \tag{2.13}$$

is a friend of $\mathcal{R}$, i.e., $F \in \mathcal{F}(\mathcal{R})$.

EXAMPLE 2.4. *Consider*

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= x_1 + u_2. \end{aligned}$$

*We will show that $V = span\{e_1, e_2\}$ is a reachability subspace.*

*It is easy to compute that* $F = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$ *is a friend of* $V$, *and* $ImB \cap V = span\{e_2\}$. *Then, it is easy to calculate that* $(A+BF)e_2 = e_1$ *and* $(A+BF)e_1 = e_1$. *Thus* $< A+BF|ImB \cap V >= V$. *We note that* $V_1 = span\{e_1\}$ *is an (A,B)-invariant subspace but not a reachability subspace since* $ImB \cap V_1 = 0$.

The next theorem shows that the representation (2.13) is independent of the actual choice of $F \in \mathcal{F}(\mathcal{R})$.

THEOREM 2.4. *Let* $\mathcal{R}$ *be a reachability subspace and let* $\hat{F} \in \mathcal{F}(\mathcal{R})$, *i.e. an arbitrary friend of* $\mathcal{R}$. *Then*

$$\mathcal{R} = \langle A + B\hat{F} | \operatorname{Im} B \cap \mathcal{R} \rangle.$$

PROOF
From Theorem 2.3 follows the existence of an $F$ such that

$$\mathcal{R} = \langle A + BF | \operatorname{Im} B \cap \mathcal{R} \rangle.$$

Now let $\hat{F} \in \mathcal{F}(\mathcal{R})$ be an arbitrary friend and form

$$\hat{\mathcal{R}} = \langle A + B\hat{F} | \operatorname{Im} B \cap \mathcal{R} \rangle.$$

Since $(A + B\hat{F})\mathcal{R} \subseteq \mathcal{R}$, it holds that $\hat{\mathcal{R}} \subseteq \mathcal{R}$.

We shall show that $R \subseteq \hat{\mathcal{R}}$ by induction. Clearly $\operatorname{Im} B \cap \mathcal{R} \subseteq \hat{\mathcal{R}}$. Assume that

$$(A + BF)^k (\operatorname{Im} B \cap \mathcal{R}) \subseteq \hat{\mathcal{R}}.$$

Then

$$\begin{aligned} (A+BF)^{k+1}(\operatorname{Im} B \cap \mathcal{R}) &\subseteq (A+BF)\hat{\mathcal{R}} \\ &\subseteq (A+B\hat{F})\hat{\mathcal{R}} + B(F-\hat{F})\hat{\mathcal{R}} \\ &\subseteq \hat{\mathcal{R}} \text{ if } B(F-\hat{F})\hat{\mathcal{R}} \subseteq \hat{\mathcal{R}} \end{aligned}$$

If we can show the last inclusion, then it follows by induction that $R \subseteq \hat{\mathcal{R}}$.

We need to introduce a lemma here in order to carry on the proof.

LEMMA 2.5. *Let* $F_1 \in \mathcal{F}(\mathcal{V})$. *Then* $F_2 \in \mathcal{F}(\mathcal{V})$ *if and only if* $B(F_1 - F_2)\mathcal{V} \subseteq \mathcal{V}$.

PROOF (Proof of the lemma)
(only if) Suppose $F_1, F_2 \in \mathcal{F}(\mathcal{V})$. Then for all $v \in \mathcal{V}$ it holds that $(A + BF_1)v \in \mathcal{V}$ and $(A + BF_2)v \in \mathcal{V}$, which implies that $B(F_1 - F_2)v \in \mathcal{V}$.
(if) Let $v \in \mathcal{V}$. Then $(A + BF_1)v + B(F_2 - F_1)v = (A + BF_2)v \in \mathcal{V}$, since the terms on the left hand side are in $\mathcal{V}$. ∎

Now we return to the proof of the theorem. Since $\hat{\mathcal{R}} \subseteq \mathcal{R}$ it holds that

$$B(F - \hat{F})\hat{\mathcal{R}} \subseteq B(F - \hat{F})\mathcal{R} \subseteq \{\text{Lemma 2.5}\} \subseteq \mathcal{R}.$$

But $B(F-\hat{F})\hat{\mathcal{R}} \subseteq \mathrm{Im}\ B$ and therefore

$$B(F-\hat{F})\hat{\mathcal{R}} \subseteq \mathrm{Im}\ B \cap \mathcal{R} \subseteq \hat{\mathcal{R}}.$$

∎

Combining Theorem 2.3 and Theorem 2.4 we obtain the following result, which can be used to test whether a given subspace is a reachability subspace.

COROLLARY 2.6. *Suppose $\mathcal{V}$ is $(A,B)$-invariant and let $F \in \mathcal{F}(\mathcal{V})$ be an arbitrary friend of $\mathcal{V}$. The necessary and sufficient condition for $\mathcal{V}$ to be a reachability subspace is that*

$$\langle A + BF | \,\mathrm{Im}\ B \cap \mathcal{V}\rangle = \mathcal{V}.$$

.

## 2.4. Maximal reachability subspaces

Consider the class $S(\mathcal{Z})$ of $(A,B)$-invariant subspaces contained in $\mathcal{Z}$, and in particular reachability subspaces $\mathcal{R}$ such that $\mathcal{R} \in S(\mathcal{Z})$. All these satisfy

$$\mathcal{R} \subseteq \mathcal{S}^*(\mathcal{Z}), \tag{2.14}$$

where $\mathcal{S}^*(\mathcal{Z})$ is the maximal $(A,B)$-invariant subspace in $\mathcal{Z}$. The existence of $\mathcal{S}^*$ is shown as follows.

LEMMA 2.7. *Let $\mathcal{Z}$ be a subspace of $\mathbb{R}^n$. Then, the class $S(\mathcal{Z})$ of all $(A,B)$-invariant subspaces $\mathcal{S} \subseteq \mathcal{Z}$ has a maximal element $\mathcal{S}^*(\mathcal{Z})$ in the sense that*

$$\mathcal{S} \subseteq \mathcal{S}^*(\mathcal{Z}) \text{ for all } \mathcal{S} \in S(\mathcal{Z}).$$

PROOF
Note first that the set $S(\mathcal{Z})$ is closed under addition, i.e., if $\mathcal{S}_1, \mathcal{S}_2 \in S(\mathcal{Z})$, then $\mathcal{S}_1 + \mathcal{S}_2 \subseteq \mathcal{Z}$ and

$$A(\mathcal{S}_1 + \mathcal{S}_2) = A\mathcal{S}_1 + A\mathcal{S}_2 \subseteq \mathcal{S}_1 + \mathcal{S}_2 + \mathrm{Im}\ B.$$

Hence, $\mathcal{S}_1 + \mathcal{S}_2 \in S(\mathcal{Z})$.

Since $\mathcal{Z}$ is of finite dimension, there is an element $\mathcal{S}^* \in S(\mathcal{Z})$ of largest dimension. If $\mathcal{S} \in S(\mathcal{Z})$, then $\mathcal{S} + \mathcal{S}^* \in S(\mathcal{Z})$ and $\mathcal{S}^* \subseteq \mathcal{S} + \mathcal{S}^*$. However, $\mathcal{S}^*$ has maximal dimension and therefore, $\dim(\mathcal{S} + \mathcal{S}^*) = \dim \mathcal{S}^*$, and then, $\mathcal{S}^* = \mathcal{S} + \mathcal{S}^*$, that is, $\mathcal{S} \subseteq \mathcal{S}^*$. Thus, $\mathcal{S}^*$ is maximal in terms of subspace inclusion. ∎

Is there also a *maximal* $\mathcal{R}$ that satisfies (2.14)? *Maximal* in the sense that it contains all other such reachability subspaces.

THEOREM 2.8. *Let $\mathcal{S}^*$ be the maximal $(A,B)$-invariant subspace in $\mathcal{Z}$, and let $F \in \mathcal{F}(\mathcal{S}^*)$. Then the maximal reachability subspace in $\mathcal{Z}$ is*

$$\mathcal{R}^* := \langle A + BF | \,\mathrm{Im}\ B \cap \mathcal{S}^*\rangle. \tag{2.15}$$

*Moreover,* $F \in \mathcal{F}(\mathcal{R}^*)$, *i.e.,*

$$\mathcal{F}(\mathcal{R}^*) \supseteq \mathcal{F}(\mathcal{S}^*).$$

We develop the proof with the help of the following two lemmas. The first of them is a refinement of Theorem 2.4, where we learn that a reachability subspace can in fact be characterized by any friend of the smaller class of friends of an $(A,B)$-invariant subspace which generates the reachability subspace in a specific way.

Lemma 2.9. *Let* $\mathcal{S}$ *be* $(A,B)$*-invariant and let*

$$\mathcal{R} := \langle A + BF |\, \hat{\mathcal{B}} \rangle,$$

*where* $F \in \mathcal{F}(\mathcal{S})$ *and* $\hat{\mathcal{B}} = \operatorname{Im} B \cap \mathcal{S}$. *If* $\hat{F}$ *is any matrix such that* $B(\hat{F} - F)\mathcal{S} \subset \mathcal{S}$ *then we also have that* $\mathcal{R} = \langle A + B\hat{F} |\, \hat{\mathcal{B}} \rangle$.

Remark 2.3. *Recalling Lemma 2.5 we see that the condition for* $\hat{F}$ *in the above lemma amounts to* $\hat{F} \in \mathcal{F}(\mathcal{S})$.

Proof
Let

$$\hat{\mathcal{R}} := \langle A + B\hat{F} |\, \hat{\mathcal{B}} \rangle$$

and

$$\mathcal{S}_i := \hat{\mathcal{B}} + (A + BF)\hat{\mathcal{B}} + \ldots + (A + BF)^{i-1}\hat{\mathcal{B}}.$$

Then $\mathcal{S}_1 \subseteq \hat{\mathcal{R}}$.

Proceeding by induction, assume that $\mathcal{S}_i \subseteq \hat{\mathcal{R}}$. Then

$$\mathcal{S}_{i+1} = \hat{\mathcal{B}} + (A + BF)\mathcal{S}_i \subseteq \hat{\mathcal{B}} + (A + B\hat{F})\mathcal{S}_i + B(F - \hat{F})\mathcal{S}_i,$$

which is included in $\hat{\mathcal{R}}$ if

$$B(F - \hat{F})\hat{\mathcal{R}} \subseteq \hat{\mathcal{R}}. \tag{2.16}$$

If so, $\mathcal{R} = \mathcal{S}_n \subseteq \hat{\mathcal{R}}$ by induction.

We now show (2.16). Since $\hat{F} \in \mathcal{F}(\mathcal{S})$ and $\hat{\mathcal{B}} \subseteq \mathcal{S}$ it follows that $\hat{\mathcal{R}} \subseteq \mathcal{S}$. Therefore,

$$B(F - \hat{F})\hat{\mathcal{R}} \subseteq B(F - \hat{F})\mathcal{S} \subseteq \hat{\mathcal{B}} \subseteq \hat{\mathcal{R}}$$

and (2.16) follows. We have thus shown that $\mathcal{R} \subseteq \hat{\mathcal{R}}$.

If we interchange $F$ and $\hat{F}$ in the calculations above, we get $\hat{\mathcal{R}} \subseteq \mathcal{R}$. ∎

Lemma 2.10. *Let* $\mathcal{R}$ *and* $\mathcal{S}$ *be* $(A,B)$*-invariant, and suppose that* $\mathcal{R} \subseteq \mathcal{S}$. *Then, if* $\hat{F} \in \mathcal{F}(\mathcal{R})$, *there is an* $F \in \mathcal{F}(\mathcal{R}) \cap \mathcal{F}(\mathcal{S})$ *such that* $F|_{\mathcal{R}} = \hat{F}|_{\mathcal{R}}$.

Proof
Let $\mathcal{W}$ be a subspace such that

$$\mathcal{R} \oplus \mathcal{W} = \mathcal{S}.$$

Let $\{w_1, \ldots, w_q\}$ be a basis for $\mathcal{W}$. Since $\mathcal{S}$ is $(A,B)$-invariant and $\mathcal{W} \subseteq \mathcal{S}$, we have

$$Aw_i = v_i + Bu_i$$

for some $v_i \in \mathcal{S}$ and $u_i \in \mathbb{R}^m$. Now let $F : \mathbb{R}^n \to \mathbb{R}^m$ be such that $Fx = \hat{F}x$ for $x \in \mathcal{R}$ and $Fw_i = -u_i$. Then $F|_{\mathcal{R}} = \hat{F}|_{\mathcal{R}}$ and $(A + BF)\mathcal{R} \subseteq \mathcal{R} \subseteq \mathcal{S}$. Moreover, $(A + BF)w_i = v_i$, i.e., $(A + BF)\mathcal{W} \subseteq \mathcal{S}$. Hence, $(A + BF)\mathcal{S} \subseteq \mathcal{S}$. ∎

We now prove Theorem 2.8.

Proof

We need to show that $\mathcal{R}^*$ as defined by $\langle A + BF|\,\mathrm{Im}\ B \cap \mathcal{S}^*\rangle$ where $F$ is any friend of $\mathcal{S}^*$, is a reachability subspace in $\mathcal{Z}$, and moreover that it is maximal.

Since $\mathrm{Im}\ B \cap \mathcal{S}^* \subseteq \mathcal{S}^*$ we have

$$\mathcal{R}^* \subseteq \langle A + BF|\,\mathcal{S}^*\rangle = \mathcal{S}^* \subseteq \mathcal{Z}$$

and we can always choose $G$ such that $\mathrm{Im}\ BG = \mathrm{Im}\ B \cap \mathcal{S}^*$. So $\mathcal{R}^*$ is a reachability subspace in $Z$.

Next we show that $\mathcal{R} \subset \mathcal{R}^*$ for all reachability subspaces contained in $\mathcal{Z}$. If $\mathcal{R}$ is an arbitrary reachability subspace in $\mathcal{Z}$, it can be expressed as

$$\mathcal{R} = \langle A + BF_0|\,\mathrm{Im}\ B \cap \mathcal{R}\rangle$$

for some $F_0 \in \mathcal{F}(\mathcal{R})$. Clearly, $\mathcal{R} \subseteq \mathcal{S}^*$. Moreover, by Lemma 2.10 there is an $F_1 \in \mathcal{F}(S^*)$ such that

$$F_1|_{\mathcal{R}} = F_0|_{\mathcal{R}}. \tag{2.17}$$

Now if $x \in \mathcal{S}^*$ then

$$B(F - F_1)x = (A + BF)x - (A + BF_1)x \in \mathcal{S}^*,$$

since $F, F_1 \in \mathcal{F}(\mathcal{S}^*)$. Hence,

$$B(F - F_1)\mathcal{S}^* \subseteq \mathrm{Im}\ B \cap \mathcal{S}^*. \tag{2.18}$$

Consequently,

$$\begin{aligned}
\mathcal{R} &= \langle A + BF_0|\,\mathrm{Im}\ B \cap \mathcal{R}\rangle && (2.19)\\
&= \langle A + BF_1|\,\mathrm{Im}\ B \cap \mathcal{R}\rangle && (2.20)\\
&\subseteq \langle A + BF_1|\,\mathrm{Im}\ B \cap \mathcal{S}^*\rangle && (2.21)\\
&= \langle A + BF|\,\mathrm{Im}\ B \cap \mathcal{S}^*\rangle = \mathcal{R}^*, && (2.22)
\end{aligned}$$

where (2.20) follows from (2.17), (2.21) follows from $\mathcal{R} \subseteq \mathcal{S}^*$, and (2.22) follows from Lemma 2.9 and (2.18). But $\mathcal{R}$ is arbitrary, and therefore $\mathcal{R}^*$ is the unique maximal reachability subspace in $\mathcal{Z}$. ∎

We conclude this section with an example.

Example 2.5. *Compute $\mathcal{R}^*$ contained in $\mathcal{Z} = ker\ C$ for*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

*For this purpose, we need to compute* $\mathcal{S}^*$ *first. Set* $\mathcal{S}_0 = \ker C$*, i.e.,*

$$S_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

*and we check if*

$$AS_0 \subset S_0 + ImB.$$

*Since*

$$[S_0, B] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

*has full dimension, clearly,*

$$AS_0 \subset S_0 + ImB.$$

*Hence,*

$$\mathcal{S}^* = \ker C = \mathrm{Im} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = [v_1, v_2].$$

*(What should we do if* $S_0 \neq \mathcal{S}^*$*?) The next step is to determine a friend for* $\mathcal{S}^*$*. Since* $(A + BF)\mathcal{S}^* \subseteq \mathcal{S}^*$ *implies*

$$A\mathcal{S}^* \subseteq \mathcal{S}^* + B(-F)\mathcal{S}^*,$$

*Therefore, we form*

$$A[v_1, v_2] = \begin{bmatrix} 2 & 0 \\ 3 & 0 \\ 2 & 3 \end{bmatrix} \tag{2.23}$$

$$= \begin{bmatrix} 0 & 0 \\ 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}. \tag{2.24}$$

*The first term in* (2.24) *is in* $\mathcal{S}^*$ *and the second term has the form* $B[u_1, u_2]$*. Hence, to find* $F$ *we must solve the system*

$$-F[v_1, v_2] = [u_1, u_2],$$

*i.e.,*

$$\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \text{ with solution } \begin{bmatrix} f_{11} & 0 & 0 \\ f_{21} & -2 & 0 \end{bmatrix}.$$

*If we choose* $f_{11} = f_{21} = 0$ *then* $A + BF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$*. Finally, straightforward computations yields*

$$\mathrm{Im}\ B \cap \mathcal{S}^* = \mathrm{Im} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \mathcal{R}^*.$$

## 2.5. Reachability under state constraints

Consider the system

$$\dot{x} \quad = \quad Ax + Bu \tag{2.25}$$

In this section we shall answer the following question. Let $\mathcal{Z}$ be an arbitrary subspace of $\mathbb{R}^n$. Which states can be reached from the origin if we require that the trajectory lies in $\mathcal{Z}$?

Before giving the main result we state some lemmas.

LEMMA 2.11. *Let $x(t,u)$ be the solution of a controlled differential equation and let $\mathcal{M}$ be a subspace of $\mathbb{R}^n$. If $x(t,u) \in \mathcal{M}$ for all $t$ then $\dot{x}(t,u) \in \mathcal{M}$ for all $t$.*

The proof is left as an exercise for the reader.

LEMMA 2.12. *Consider the system* (2.25) *and let $\mathcal{Z}$ be a subspace of $\mathbb{R}^n$. If $x(t) \in \mathcal{Z}$ for $t \geq 0$ then $x(t) \in \mathcal{S}^*(\mathcal{Z})$ for $t \geq 0$.*

The proof is left as an exercise for the reader.

LEMMA 2.13. *Consider the system* (2.25) *and let $\mathcal{Z}$ be a subspace of $\mathbb{R}^n$. If $x(0) = 0$ and $x(t) \in \mathcal{Z}$ for $t \geq 0$ then $x(t) \in \mathcal{R}^*(\mathcal{Z})$ for $t \geq 0$.*

PROOF
By Lemma 2.12 we know that $x(t) \in \mathcal{S}^*(\mathcal{Z})$ for $t \geq 0$. Now, let $F$ be a friend of $\mathcal{S}^*(\mathcal{Z})$ and write the input as

$$u = Fx + v.$$

Then

$$Bv(t) = \dot{x}(t) - (A + BF)x(t) \in \mathcal{S}^*(\mathcal{Z}) \text{ for } t \geq 0,$$

by Lemma 2.11 and Lemma 2.12. Hence,

$$Bv(t) \in \text{Im } B \cap \mathcal{S}^*(\mathcal{Z}),$$

which implies that

$$x(t) = \int_0^t e^{(A+BF)(t-s)} Bv(s)\, ds \in \langle A + BF |\ \text{Im } B \cap \mathcal{S}^*(\mathcal{Z})\rangle = \mathcal{R}^*(\mathcal{Z}).$$

∎

We can now solve the problem of reachability under constraints.

THEOREM 2.14. *Let $\mathcal{S}$ be the set of states that can be reached from the origin with trajectories in $\mathcal{Z}$, i.e.,*

$$S := \{x \in \mathcal{Z} \,|\, \exists\, t_1 : x(t_1) = x \text{ and } x(t) \in \mathcal{Z}\, \forall t \in [0, t_1]\}. \tag{2.26}$$

*Then,*

$$\mathcal{R}^*(\mathcal{Z}) = S.$$

Proof

We first show that $\mathcal{S} \subseteq \mathcal{R}^*(\mathcal{Z})$. Let $x \in \mathcal{S}$. Then there is an input and a time $t_1$ such that $x(t) \in \mathcal{Z}$ and $x(t_1) = x$. By Lemma 2.13 it follows that $x(t) \in \mathcal{R}^*(\mathcal{Z})$ for $t \leq t_1$, and in particular, $x(t_1) \in \mathcal{R}^*(\mathcal{Z})$.

Next we show that $\mathcal{R}^*(\mathcal{Z}) \subseteq \mathcal{S}$. Let $F$ and $G$ be such that $\mathcal{R}^*(\mathcal{Z})$ is the reachable subspace of the closed-loop system

$$\dot{x} = (A + BF)x + BGv.$$

Clearly, all points in $\mathcal{R}^*(\mathcal{Z})$ can be reached by trajectories in $\mathcal{R}^*(\mathcal{Z})$. Hence, $\mathcal{R}^*(\mathcal{Z}) \subseteq \mathcal{S}$. ∎