Mathematical Systems Theory: Advanced Course
Exercise Session 1

1 Linear algebra

Let $\mathcal{X}$ and $\mathcal{Y}$ be linear vector spaces over R, and A is a map from $\mathcal{X}$ to $\mathcal{Y}$.

- Subspace $\mathcal{S}$ in $\mathcal{X}$: $\mathcal{S} \subset \mathcal{X}$ and

  $$\alpha_1 s_1 + \alpha_2 s_2 \in \mathcal{S}, \forall \alpha_1, \alpha_2 \in R, \text{ and } \forall s_1, s_2 \in \mathcal{S}.$$

  For R^n, subspaces $\Leftrightarrow$ hyperplanes passing through origin.

- Image space

  $$\text{Im } A := \{y \in \mathcal{Y} : y = Ax \text{ for some } x \in \mathcal{X}\}.$$

- Rank of A

  $$\text{rank} A := \dim(\text{Im } A),$$

  where $\dim(\text{Im } A)$ is the number of linearly independent vectors in the subspace Im A.

- Kernel space (Null space)

  $$\ker A := \{x \in \mathcal{X} : Ax = 0\}.$$

- Preimage of $\mathcal{W}(\subset \mathcal{Y})$ under the map A

  $$A^I \mathcal{W} := \{x \in \mathcal{X} : Ax \in \mathcal{W}\}.$$

Example

For the following matrix A and $\mathcal{W}$, obtain Im A, rank$A$, $\ker A$ and $A^I \mathcal{W}$,

$$A := \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \ \mathcal{W} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Since Ax is calculated as

$$Ax = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \cdots = (x_1 + x_2) \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{=:v_1} + (x_2 + x_3) \underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}_{=:v_2},$$

we obtain

$$\begin{aligned}
\text{Im } A &= \{Ax : x \in R^3\} \\
&= \{(x_1 + x_2)v_1 + (x_2 + x_3)v_2,\ x_1, x_2, x_3 \in R\} \\
&= \text{span}\{v_1, v_2\}. \text{ (This expression is not unique.)} \\
\text{rank} A &= \dim(\text{Im } A) = 2. \\
\ker A &= \{x \in R^3 : Ax = 0\} \\
&= \{x \in R^3 : (x_1 + x_2)v_1 + (x_2 + x_3)v_2 = 0,\ x_1, x_2, x_3 \in R\} \\
&= \{x \in R^3 : x_1 = x_3 = -x_2\} \\
&= \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\} \\
A^I \mathcal{W} &= \{x \in R^3 : Ax \in \mathcal{W}\} \\
&= \{x \in R^3 : x_2 + x_3 = 0\} \\
&= \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} : x_1, x_2 \in R \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}
\end{aligned}$$

Important facts

- Let A be a linear map from $\mathcal{X}$ to $\mathcal{Y}$.
  - Im A is a subspace of $\mathcal{Y}$.
  - $\ker A$ is a subspace of $\mathcal{X}$.
- Let $\mathcal{S}$ and $\mathcal{T}$ be subspaces of $\mathcal{X}$. Then,
  - $\mathcal{S} + \mathcal{T}$ is a subspace of $\mathcal{X}$, where

    $$\mathcal{S} + \mathcal{T} := \{s + t :\ s \in \mathcal{S},\ t \in \mathcal{T}\}.$$

    Note that $\mathcal{S} + \mathcal{S} = \mathcal{S}$! and $\mathcal{S} - \mathcal{S} = \mathcal{S}$!
  - $\mathcal{S} \cap \mathcal{T}$ is a subspace of $\mathcal{X}$.

– $\mathcal{S} \cup \mathcal{T}$ is NOT a subspace of $\mathcal{X}$ in general.

- Let A_1 and A_2 be maps from a space $\mathcal{X}$ to a space $\mathcal{Y}$. Define a map $A_1 + A_2$ from $\mathcal{X}$ to $\mathcal{Y}$ as

$$(A_1 + A_2)x := A_1 x + A_2 x.$$

Then,

$$(A_1 + A_2)\mathcal{X} \subset A_1\mathcal{X} + A_2\mathcal{X}.$$

Problems (Linear algebra)

1. Show that for linear vector spaces $\mathcal{D}$ and $\mathcal{M}$ and a linear operator $L : \mathcal{D} \mapsto \mathcal{M}$,

   (a) the kernel of L is a subspace of $\mathcal{D}$.

   (b) the image of L is a subspace of $\mathcal{M}$.

   (c) if $\mathcal{D} = \mathcal{M}$ the image of L is an L-invariant subspace of $\mathcal{M}$.

   (d) any space spanned by a subset of the eigenvectors of L is an L-invariant subspace of $\mathcal{M}$.

2. Let $\mathcal{R}, \mathcal{S}, \mathcal{T}$ be subspaces of $\mathcal{X}$, and suppose $\mathcal{S} \subset \mathcal{R}$. Show that

$$\mathcal{R} \cap (\mathcal{S} + \mathcal{T}) = \mathcal{R} \cap \mathcal{S} + \mathcal{R} \cap \mathcal{T} = \mathcal{S} + \mathcal{R} \cap \mathcal{T}.$$

   Note that the intersection is not distributive in general. Consider the case of three one-dimensional subspaces of the plane for a simple counterexample.

   The following subspace inclusion holds without the assumption $\mathcal{S} \subset \mathcal{R}$,

$$\mathcal{R} \cap (\mathcal{S} + \mathcal{T}) \supset \mathcal{R} \cap \mathcal{S} + \mathcal{R} \cap \mathcal{T},$$

   and it is easy to prove.

3. Suppose $\mathcal{X}$ is a vector space, $\mathcal{V}, \mathcal{W} \subset \mathcal{X}$ are subspaces, and $A : \mathcal{X} \to \mathcal{X}$ linear. Give proofs or counterexamples for the following claims. Here A^I denotes preimage of A, i.e. $A^I\mathcal{V} \triangleq \{x \in \mathcal{X} | Ax \in \mathcal{V}\}$.

   (a) $A^I\mathcal{V} \subset \mathcal{W}$ implies $\mathcal{V} \subset A\mathcal{W}$.

   (b) $\mathcal{V} \subset A\mathcal{W}$ implies $A^I\mathcal{V} \subset \mathcal{W}$

(c) $\mathcal{V} \subset \mathcal{W}$ implies $A\mathcal{V} \subset A\mathcal{W}$

(d) $\mathcal{V} \subset \mathcal{W}$ implies $A^I\mathcal{V} \subset A^I\mathcal{W}$

(e) $A(A^I\mathcal{V}) = \mathcal{V} \cap \text{Im } A$

(f) $A^I(A\mathcal{V}) = \mathcal{V} + \ker A$

(g) $A\mathcal{V} \subset \mathcal{W}$ if and only if $\mathcal{V} \subset A^I\mathcal{W}$

4. $\mathcal{V}, \mathcal{W} \subset \mathcal{X}$ are subspaces that are invariant for a linear operator $A : \mathcal{X} \to \mathcal{X}$. Give proofs or counterexamples for the following claims.

   (a) $\mathcal{V} + \mathcal{W}$ is an invariant subspace for A.

   (b) $\mathcal{V} \cup \mathcal{W}$ is an invariant subspace for A.

   (c) $\mathcal{V} \cap \mathcal{W}$ is an invariant subspace for A.

   (d) $A^I(\mathcal{V} \cap \mathcal{W})$ is an invariant subspace for A.

5. Let C be a linear mapping $C : \mathcal{X} \mapsto \mathcal{Y}$, and assume that the subspaces $\mathcal{R}_1, \mathcal{R}_2 \subset \mathcal{X}$ and $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{Y}$ are given and arbitrary. Show that

   (a) $C(\mathcal{R}_1 + \mathcal{R}_2) = C\mathcal{R}_1 + C\mathcal{R}_2$.

   (b) $C(\mathcal{R}_1 \cap \mathcal{R}_2) \subset (C\mathcal{R}_1) \cap (C\mathcal{R}_2)$.

   (c) $C^I(\mathcal{S}_1 + \mathcal{S}_2) \supset C^I\mathcal{S}_1 + C^I\mathcal{S}_2$.

   (d) $C^I(\mathcal{S}_1 \cap \mathcal{S}_2) = (C^I\mathcal{S}_1) \cap (C^I\mathcal{S}_2)$.

6. Let C_1, C_2 be linear mappings $C_i : \mathcal{X} \mapsto \mathcal{Y}$ and let $\mathcal{R} \subset \mathcal{X}$ be an arbitrary subspace. Define a sum of two mappings $C_1 + C_2 : \mathcal{X} \mapsto \mathcal{Y}$ by $(C_1 + C_2)x := C_1x + C_2x$. Show that

   (a) $(C_1 + C_2)\mathcal{R} \subset C_1\mathcal{R} + C_2\mathcal{R}$,

   (b) $(C_1 + C_2)\mathcal{R} = (C_1 - C_2)\mathcal{R}$ does not generally hold.

7. Show that if $\mathcal{U}$ and $\mathcal{V}$ are finite dimensional subspaces of $\mathcal{W}$ and $\dim\mathcal{U} + \dim\mathcal{V} > \dim\mathcal{W}$ then $\mathcal{U} \cap \mathcal{V} \neq \{0\}$.

2 Invariant subspaces

Let $\mathcal{S}$ be a subspace in R^n, A be a linear map from R^n to R^n and B be a linear map from R^m to R^n.

A-invariant subspaces

- $\mathcal{S}$ is A-*invariant* (*invariant subspace* under A) if

$$A\mathcal{S} \subset \mathcal{S}.$$

How to check this? There are two different ways to check if a set is A-invariant.

Method 1

Although $\mathcal{S}$ has infinite elements, we do *not* need to check $As \in \mathcal{S}$ for each $s \in \mathcal{S}$, and we have only to check $Av_j \in \mathcal{S}$ for the basis $\{v_j\}_{j=1}^{p}$ of the subspace $\mathcal{S}$.

Suppose that $Av_j \in \mathcal{S}$ for $j = 1, \ldots, p$. Any element in $\mathcal{S}$ can be expressed as $\sum_{j=1}^{p} \alpha_j v_j$.

$$\begin{aligned} A\left(\sum_{j=1}^{p} \alpha_j v_j\right) &= \sum_{j=1}^{p} \alpha_j A v_j \text{ (since } A \text{ is linear)} \\ &\in \mathcal{S} \text{ (since } Av_j \in \mathcal{S} \text{ and } \mathcal{S} \text{ is a subspace)} \end{aligned}$$

Therefore, it is enough to check if $Av_j \in \mathcal{S}$ for $j = 1, \ldots, p$.

Suppose now that $Av_j \in \mathcal{S}$ for $j = 1, \ldots, p$. Then, we can write Av_j as

$$Av_j = \sum_{i=1}^{p} \beta_{ij} v_i, \; j = 1, \ldots, p.$$

In a matrix form,

$$\begin{aligned} A \underbrace{\begin{bmatrix} v_1 & \cdots & v_p \end{bmatrix}}_{=:V} &= \begin{bmatrix} \sum_{i=1}^{p} \beta_{i1} v_i & \cdots & \sum_{i=1}^{p} \beta_{ip} v_i \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} v_1 & \cdots & v_p \end{bmatrix}}_{=:V} \underbrace{\begin{bmatrix} \beta_{11} & \cdots & \beta_{1p} \\ \vdots & \ddots & \vdots \\ \beta_{p1} & \cdots & \beta_{pp} \end{bmatrix}}_{K}. \end{aligned}$$

Conversely, if we can transform AV into a form VK, then we can conclude that $A\mathcal{S} \subset \mathcal{S}$.

To recap, in order to check if $\mathcal{S}$ is A-invariant, try to find a matrix K satisfying

$$AV = VK,$$

where V consists of the basis of $\mathcal{S}$. If this is possible (impossible), $\mathcal{S}$ is A-invariant (not A-invariant).

Method 2

Another way to check if a set S is A-invariant is to define S via

$$S := \{x \in \mathcal{R}^n : Px = 0\}.$$

Then $Ax \in S$, $\forall x \in S$ implies $PAx = 0 \Rightarrow P\dot{x} = 0$, $\forall x \in S$.

(A,B)-invariant subspaces

- $\mathcal{S}$ is an (A, B)-*invariant (controlled invariant) subspace* if there exists an F satisfying
$$(A + BF)\mathcal{S} \subset \mathcal{S}.$$

Method 1

A necessary and sufficient condition for a subspace $\mathcal{S}$ to be (A, B)-invariant is (see Theorem 2.2 in page 11 in the lecture note)

$$A\mathcal{S} \subset \mathcal{S} + \text{Im } B.$$

Note that this condition does not involve F. To check this condition, again we have only to check

$$Av_j \in \mathcal{S} + \text{Im } B, \; j = 1, \ldots, p.$$

In this case,

$$Av_j = \sum_{i=1}^{p} \beta_{ij} v_i + Bu_j, \; j = 1, \ldots, p,$$

or in a matrix form,

$$A \begin{bmatrix} v_1 & \cdots & v_p \end{bmatrix} = VK + B \underbrace{\begin{bmatrix} u_1 & \cdots & u_p \end{bmatrix}}_{U}.$$

Conversely, if we can rewrite AV as the form $VK+BU$, then $A\mathcal{S} \subset \mathcal{S}+\text{Im } B$ holds.

To recap, in order to check if $\mathcal{S}$ is (A, B)-invariant, try to find matrices K and U satisfying

$$AV = VK + BU,$$

where V consists of the basis of $\mathcal{S}$. If this is possible (impossible), $\mathcal{S}$ is (A, B)-invariant (not (A, B)-invariant).

Now, the question is

- How to find a friend F which makes $\mathcal{S}$ to be $(A + BF)$-invariant?

You need to solve $FV = -U$ (see also the lectures notes, Section 2.2).

Method 2

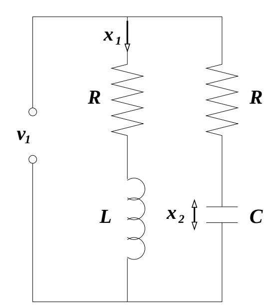
To check if a set S is (A,B)-invariant, we can define S via

$$S := \{x \in \mathcal{R}^n : Px = 0\}.$$

Then we try to find a state feedback $u = Fx$ such that $(A + BF)x \in S$, $\forall x \in S$ implies $P(A + BF)x = 0 \Rightarrow P\dot{x} = 0$, $\forall x \in S$.

Example (Invariant subspace)

Consider the following circuit system (which we took from the lecture note of *Matematisk systemteori grundkurs*).



This system has a state space description as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & 0 \\ 0 & -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \frac{1}{RC} \end{bmatrix} v_1.$$

Now, we suppose that the input signal u_1 is a sinusoidal with some additional term

$$\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Ignoring the physical reasonability, we assume $R = L = C = \omega = 1$. Then the system can be written in the following way.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=:B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

In order to not get confused in the notations, we change the variables v_1, v_2 to x_3, x_4, we get the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_{=:A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{=:B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

First, we suppose that there is no input, i.e., $u_1 = u_2 = 0$. Let us consider the following two subspaces.

1. $\mathcal{S}_1 := \text{span}\{e_1, e_2\}$.

   Is $\mathcal{S}_1$ A-invariant?
   (Method 1). Since

   $$A \begin{bmatrix} e_1 & e_2 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \end{bmatrix} (-I),$$

   $A\mathcal{S}_1 \subset \mathcal{S}_1 \Rightarrow \mathcal{S}_1$ is A-invariant.

   Alternatively *(Method 2)*,

   $$S_1 = \{x \in \mathcal{R}^n : x_3 = 0, x_4 = 0\}.$$

S_1 A-invariant

$$\Rightarrow \left\{ \begin{array}{l} \dot{x}_3 = 0 \\ \dot{x}_4 = 0 \end{array} \right. \quad \forall x \in S_1 \Rightarrow \left\{ \begin{array}{l} x_4 = 0 \\ -x_3 = 0 \end{array} \right. \quad \forall x \in S_1$$

So S_1 is A-invariant.

2. $\mathcal{S}_2 := \text{span}\,\{e_2, e_4\}$.
Is $\mathcal{S}_2$ A-invariant?
(Method 1). Since

$$A \left[\begin{array}{cc} e_2 & e_4 \end{array} \right] = \left[\begin{array}{cc} -e_2 & e_3 \end{array} \right],$$

$A\mathcal{S}_2$ is NOT a subspace of $\mathcal{S}_2 \Rightarrow \mathcal{S}_2$ is NOT A-invariant.
Then, the question is

- Is it possible to use a state feedback $u = Fx$ so that $\mathcal{S}_2$ becomes $(A + BF)$-invariant?

The possibility can be checked by testing if $\mathcal{S}_2$ is (A, B)-invariant.

$$A \left[\begin{array}{cc} e_2 & e_4 \end{array} \right] = \left[\begin{array}{cc} -e_2 & e_3 \end{array} \right] = \left[\begin{array}{cc} e_2 & e_4 \end{array} \right] \left[\begin{array}{cc} -1 & 0 \\ 0 & 0 \end{array} \right] + B \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right],$$

and hence $A\mathcal{S}_2 \subset \mathcal{S}_2 + \text{Im } B \Rightarrow \mathcal{S}_2$ is (A, B)-invariant.

What are the F that make $\mathcal{S}_2$ to be $(A + BF)$-invariant?
Let's solve

$$FV = -U$$

$$\left[\begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \\ f_5 & f_6 & f_7 & f_8 \end{array} \right] \left[\begin{array}{cc} e_2 & e_4 \end{array} \right] = \left[\begin{array}{cc} 0 & -1 \\ 0 & 0 \end{array} \right]$$

So, F is of the form ($\star$=anything)

$$F_1 = \left[\begin{array}{cccc} \star & 0 & \star & -1 \\ \star & 0 & \star & 0 \end{array} \right]$$

Alternatively *(Method 2)*,

$$S_2 = \{x \in \mathcal{R}^n : x_1 = 0, x_3 = 0\}.$$

S_2 A-invariant

$$\Rightarrow \begin{cases} \dot{x}_1 = 0 \\ \dot{x}_3 = 0 \end{cases} \forall x \in S_2 \Rightarrow \begin{cases} -x_1 + x_3 = 0 \\ x_4 = 0 \end{cases} \forall x \in S_2 \quad \textit{Not true!}$$

So, S_2 is NOT A-invariant.
S_2 controlled invariant?

$$\Rightarrow \begin{cases} \dot{x}_1 = 0 \\ \dot{x}_3 = 0 \end{cases} \forall x \in S_2 \Rightarrow \begin{cases} -x_1 + x_3 = 0 \\ x_4 + u_1 = 0 \end{cases} \forall x \in S_2$$

Let $u_1 = f_1 x_1 + f_2 x_2 + f_3 x_3 + f_4 x_4$, with $f_4 = -1$, $f_2 = 0$, f_1, f_3 arbitrary, $u_2 = f_5 x_1 + f_6 x_2 + f_7 x_3 + f_8 x_4$ arbitrary, then we have $P\dot{x} \in S_2$ $\forall x \in S_2$, so S_2 is controlled invariant.
Here, F is of the form

$$F_2 = \begin{bmatrix} \star & 0 & \star & -1 \\ \star & \star & \star & \star \end{bmatrix}$$

Remark Note that the friend F_2 obtained from the second method is more general than F_1 obtained from the first method.

Problem (Invariant subspaces)

For the following A, B and $\mathcal{S}$, check if $\mathcal{S}$ is A-invariant, and if $\mathcal{S}$ is (A, B)-invariant. Try to find a friend F of $\mathcal{S}$ (if $\mathcal{S}$ is (A, B)-invariant.)

1. $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathcal{S} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

2. $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathcal{S} = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

3. $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathcal{S} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

4. $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathcal{S} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}$