Mathematical Systems Theory: Advanced Course
Exercise Session 4

1 Solving Sylvester equation

In control theory, we often need to solve Sylvester equations, as can be seen in Chapters 6 & 7 in the lecture notes. Here, we will discuss the solvability of the equation and explain how to solve it.

Suppose that $A \in R^{n\times n}$, $B \in R^{m\times m}$ and $C \in R^{n\times m}$ are given. The Sylvester equation is defined as

$$AX - XB = C.$$

Solvability condition

The Sylvester equation has a unique solution $X \in R^{n\times m}$ if and only if

$$\sigma(A) \cap \sigma(B) = \emptyset,$$

where $\sigma(\cdot)$ denotes the spectrum (i.e., the set of all eigenvalues) of a matrix.
Note. An important special case of the Sylvester equation is the *Lyapunov equation*:

$$AX + XA^T = C,$$

namely, $B := -A^T$.

How to solve the Sylvester equation?

The Sylvester equation is a linear equation, and hence, it is not difficult to solve. Indeed, we can rewrite the equation as

$$(I_m \otimes A - B^T \otimes I_n)\text{vec } X = \text{vec } C,$$

where $\otimes$ means the Kronecker product and vec X is the column expansion of a matrix X. (But you do not need this transformation in this course.)

In some simple cases, we can solve the Sylvester equation by means of direct calculations. See Examples below.
Note. In MATLAB, the command `lyap.m` solves the Sylvester equation.

2 Output tracking input

Consider a SISO system:

$$(\Sigma) \begin{cases} \dot{x} &= Ax + bu \\ y &= cx, \end{cases}$$

where A is a stable matrix, and an exosystem:

$$\begin{cases} \dot{w} &= \Gamma w \\ u &= qw, \end{cases}$$

where Γ is an antistable matrix. In addition, we assume that the controllable pair (A, b) and the observable pair (q, Γ) are given.

Our task is to design a vector c so that the output y tracks the input u asymptotically. Such c can be obtained by solving the following two equations. One is a Sylvester equation

$$A\Pi - \Pi\Gamma = -bq$$

with respect to Π, and the other is a linear equation

$$c\Pi = q$$

with respect to c. (To understand why the c obtained in this way satisfies our requirement, read the discussions in page 46 & 48 of the lecture notes). Due to Theorem 6.4 in the lecture notes, such c exists if and only if $\dim A \geq \dim \Gamma$.

Example

Consider the following SISO system:

$$\begin{cases} \dot{x} &= \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}}_{=:A} x + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=:b} u, \\ y &= cx. \end{cases}$$

We will investigate, for various choices of c, how the output y tracks the sinusoidal input u:

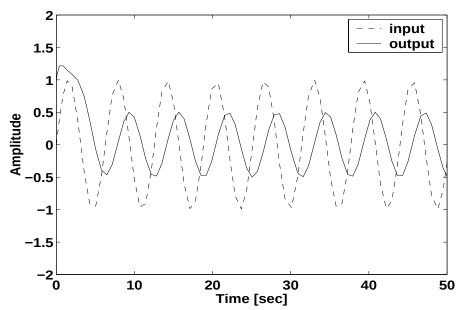
$$u(t) = r\sin(\omega t + \phi).$$

This type of input can be considered as an output of an exosystem:

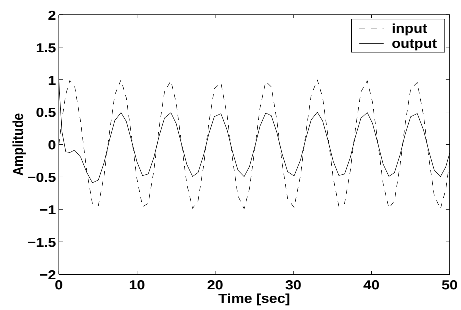
$$\left\{ \begin{array}{rcl} \begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} & = & \underbrace{\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}}_{=:\Gamma} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \quad \begin{bmatrix} w_1(0) \\ w_2(0) \end{bmatrix} = \begin{bmatrix} r\sin\phi \\ r\cos\phi \end{bmatrix} \\ u & = & \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{=:q} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \end{array} \right.$$

We set $r = \omega = 1$ and $\phi = 0$ in this example. (But we pretend that we do *not* know r and ϕ when designing c.)

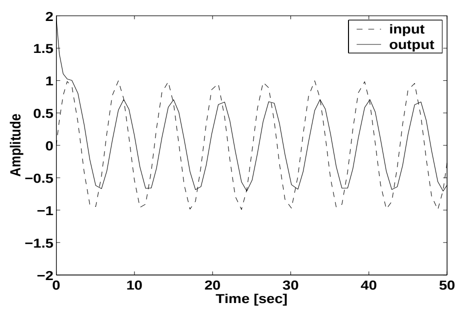
The outputs y are shown in Figure 1 for the cases where $c = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $c = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $c = \begin{bmatrix} 0 & 2 \end{bmatrix}$. (We set the initial state as $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.) As can be seen in the figure, the output y tracks the input u asymptotically in the case where $c = \begin{bmatrix} 0 & 2 \end{bmatrix}$.



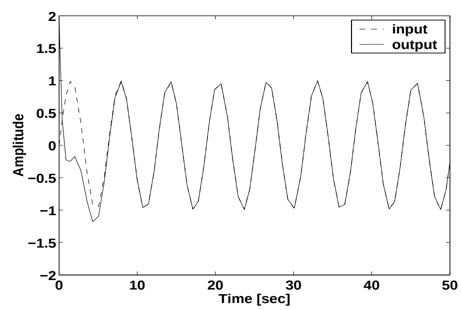
(a) $c = (1, 0)$



(b) $c = (0, 1)$



(c) $c = (1, 1)$



(d) $c = (0, 2)$

Figure 1: Output y for various c

Now, a question is how to design a "best" c, in the sense that the output y tracks the input u asymptotically. Due to Proposition 6.2 and the discussion in page 48, we first solve a Sylvester equation

$$A\Pi - \Pi\Gamma = -bq,$$

with respect to Π, and then solve a linear equation

$$c\Pi = q,$$

with respect to c. The latter equation is solvable since $\dim A \geq \dim \Gamma$ (see Theorem 6.4 in the lecture notes).

The Sylvester equation $A\Pi - \Pi\Gamma = -bq$ can be solved as follows.

$$\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}}_{\Pi} - \underbrace{\begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}}_{\Pi} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\Gamma} = -\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{q}$$

$$\Leftrightarrow \begin{cases} \pi_{21} + \pi_{12} = 0 \\ \pi_{22} - \pi_{11} = 0 \\ -\pi_{11} - 2\pi_{21} + \pi_{22} = -1 \\ -\pi_{12} - 2\pi_{22} - \pi_{21} = 0 \end{cases}$$

$$\Leftrightarrow \Pi = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Then, the equation $c\Pi = q$ is solved as

$$\underbrace{\begin{bmatrix} c_1 & c_2 \end{bmatrix}}_{c} \underbrace{\frac{1}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\Pi} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{q}$$

$$\Rightarrow \begin{cases} c_2/2 = 1 \\ -c_1/2 = 0 \end{cases}$$

$$\Rightarrow c = \begin{bmatrix} c_1 & c_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix}.$$

The obtained c coincides with the best c in the simulation result above.

3 Error feedback output regulation

Consider a MIMO system

$$\begin{cases} \dot{x} &= Ax + Bu + Pw \\ \dot{w} &= Sw \\ e &= Cx - Qw, \end{cases}$$

with the assumptions that

- (A,B) is stabilizable,
- $\left(\begin{bmatrix} C & -Q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \right)$ is detectable, and
- S is antistable.

The *error feedback output regulation problem* is to design a dynamic controller

$$\left\{ \begin{array}{rcl} \dot{z} & = & Fz + Ge \\ u & = & Hz, \end{array} \right.$$

satisfying the following requirements:

1. The closed-loop system with $w \equiv 0$ is asymptotically stable.
2. $e(t) \to 0$ as t goes to infinity for any initial state.

Solvability conditions

Due to Theorem 7.2 (page 58), this regulation problem is solvable if and only if there exist matrices Π and Γ satisfying

$$\begin{array}{rcl} \Pi S & = & A\Pi + P + B\Gamma \\ 0 & = & C\Pi - Q. \end{array}$$

Equivalently, due to Proposition 7.4 (page 59), the regulation problem is solvable if and only if the Rosenbrock matrix

$$\begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}$$

is row full rank at each eigenvalue of S. (We do *not* need P and Q to decide the solvability of the regulation problem.)

Control u

From the sufficiency proof of Theorem 7.2 (page 58),

$$\left\{ \begin{array}{rcl} \dot{z} & = & \underbrace{\left(\begin{bmatrix} A & P \\ 0 & S \end{bmatrix} + L \begin{bmatrix} C & -Q \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} \begin{bmatrix} K & -K\Pi + \Gamma \end{bmatrix} \right)}_{=:F} z + \underbrace{(-L)}_{=:G} e \\ u & = & \underbrace{\begin{bmatrix} K & -K\Pi + \Gamma \end{bmatrix}}_{=:H} z, \end{array} \right.$$

where K and L are matrices which make

$$A + BK \quad \text{and} \quad \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} + L \begin{bmatrix} C & -Q \end{bmatrix}$$

stable respectively (note that such K and L exist due to the stabilizability and detectability assumptions), and Π and Γ are the solutions of the above equations. (To find K and L, the Matlab command `place.m` will be useful.)

Example

Consider a system

$$\begin{cases} \dot{x}_1 &= -x_1 + x_3 + 2w_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 + 3x_2 + 2x_3 + u, \end{cases}$$

influenced by an exosystem

$$\begin{cases} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= -w_1. \end{cases}$$

Our goal is to find a control u that solves the error feedback output regulation problem, with the error signal $x_1 - w_1$.

This system can be expressed in a combined form:

$$\begin{cases} \dot{x} = \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} u + \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{P} w \\ \dot{w} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{S} w \\ e = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} x - \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{Q} w. \end{cases}$$

We can verify that (A, B) is controllable and $\left(\begin{bmatrix} C & -Q \end{bmatrix}, \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \right)$ is observable by using Matlab commands `ctrb.m` and `obsv.m`.

We solve the linear matrix equations:

$$\begin{aligned} \Pi S &= A\Pi + P + B\Gamma \\ 0 &= C\Pi - Q. \end{aligned}$$

First, we obtain Π from the second equation.

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \pi_1 & \pi_2 \\ \pi_3 & \pi_4 \\ \pi_5 & \pi_6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \Longrightarrow \Pi = \begin{bmatrix} 1 & 0 \\ \pi_3 & \pi_4 \\ \pi_5 & \pi_6 \end{bmatrix}$$

Next, we solve the first equation.

$$\begin{aligned}
\underbrace{\begin{bmatrix} 1 & 0 \\ \pi_3 & \pi_4 \\ \pi_5 & \pi_6 \end{bmatrix}}_{\Pi} \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{S} &= \underbrace{\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 0 \\ \pi_3 & \pi_4 \\ \pi_5 & \pi_6 \end{bmatrix}}_{\Pi} + \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix}}_{\Gamma} \\
\begin{bmatrix} 0 & 1 \\ -\pi_4 & \pi_3 \\ -\pi_6 & \pi_5 \end{bmatrix} &= \begin{bmatrix} -1+\pi_5 & \pi_6 \\ \pi_5 & \pi_6 \\ 1+3\pi_3+2\pi_5 & 3\pi_4+2\pi_6 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ \gamma_1 & \gamma_2 \end{bmatrix} \\
\begin{bmatrix} \pi_3 & \pi_4 \\ \pi_5 & \pi_6 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\
\begin{bmatrix} \gamma_1 & \gamma_2 \end{bmatrix} &= \begin{bmatrix} -3 & -6 \end{bmatrix}
\end{aligned}$$

Using matrices K and L obtained by using `place.m`, we can get a controller

$$\begin{cases} \dot{z} = Fz + Ge \\ u = Hz. \end{cases}$$

The closed-loop system becomes

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{z} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & BH & P \\ GC & F & -GQ \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} x \\ z \\ w \end{bmatrix} \\ e = \begin{bmatrix} C & 0 & -Q \end{bmatrix} \begin{bmatrix} x \\ z \\ w \end{bmatrix} \end{cases}$$

For various initial states of $\begin{bmatrix} x & w & z \end{bmatrix}^T$, the error signals e are shown in Figure 2. We can see in the figure that the error signals converge to zero irrespective of initial states.

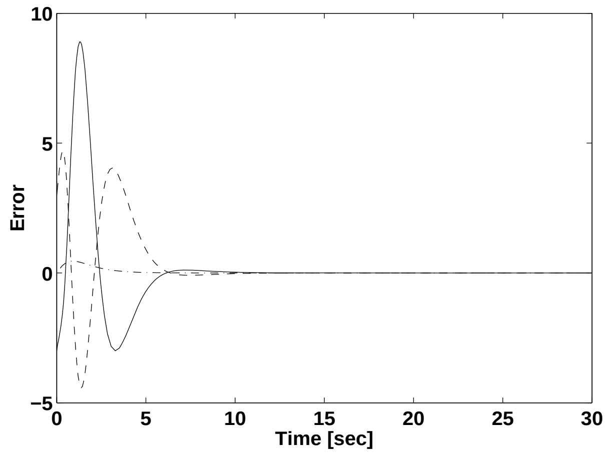


Figure 2: Error signals e for various initial states

In the case of a normal form (Important!)

If (A, B, C) are already in a normal form, the equations

$$\begin{cases} \Pi S &= A\Pi + P + B\Gamma \\ 0 &= C\Pi - Q \end{cases}$$

can be solved efficiently. To illustrate this, we consider a scalar case (analogous discussion can be done in MIMO cases) and

$$A := \begin{bmatrix} N & \hat{P} \\ \begin{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \\ \hat{R} \end{bmatrix} & \begin{bmatrix} \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix} \\ \hat{S} \end{bmatrix} \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ L \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}.$$

We divide the matrices Π and P as

$$\Pi =: \begin{bmatrix} \pi_z \\ \pi_1 \\ \vdots \\ \pi_r \end{bmatrix}, \quad P =: \begin{bmatrix} p_z \\ p_1 \\ \vdots \\ p_r \end{bmatrix}.$$

Then, from the equation $0 = C\Pi - Q$, we obtain

$$\pi_1 = Q.$$

From the equation $\Pi S = A\Pi + P + B\Gamma$, we obtain

$$\begin{array}{lll} \pi_1 S = \pi_2 + p_1 & \Rightarrow & \pi_2 = \pi_1 S - p_1 \\ \pi_2 S = \pi_3 + p_2 & \Rightarrow & \pi_3 = \pi_2 S - p_2 \\ & \vdots & \\ \pi_{r-1} S = \pi_r + p_{r-1} & \Rightarrow & \pi_r = \pi_{r-1} S - p_{r-1} \\ \pi_z S = N\pi_z + \hat{P} \begin{bmatrix} \pi_1 \\ \vdots \\ \pi_r \end{bmatrix} + p_z & \Rightarrow & \pi_z (\text{by solving the Sylvester equation}) \end{array}$$

So, we have obtained Π, *without using* Γ.

Finally, to obtain Γ, from the last row of $\Pi S = A\Pi + P + B\Gamma$,

$$\pi_r S = \begin{bmatrix} \hat{R} & \hat{S} \end{bmatrix} \Pi + p_r + L\Gamma,$$

from which we can obtain

$$\Gamma = L^{-1} \left(\pi_r S - \begin{bmatrix} \hat{R} & \hat{S} \end{bmatrix} \Pi - p_r \right).$$