Mathematical Systems Theory: Advanced Course
Exercise Session 5

1 Accessibility of a nonlinear system

Consider an affine nonlinear control system:

$$\dot{x} = f(x) + G(x)u,\ x(0) = x_0, \quad G(x) = \left[\begin{array}{ccc} g_1(x) & \cdots & g_m(x) \end{array}\right],$$

where $x \in N \subset R^n$, N is an open set and $u \in R^m$. We will discuss the accessibility of this system, which is a weaker concept than the controllability.

Definition

The system is called *locally strongly accessible from* x_0 if for any initial point in the neighborhood of x_0, the set of reachable points with appropriate u contains a non-empty open set for any sufficiently small final time T.

Proposition

If $\dim \mathcal{R}_c(x_0) = n$, then the system is locally strongly accessible from x_0, where $R_c(x)$ is the *strong accessibility distribution* (see page 67 in the lecture note).

Procedure to compute $\mathcal{R}_c(x)$

Step 1. Take

$$\mathcal{R}_0(x) = \operatorname{span}\{g_1(x), \cdots, g_m(x)\}.$$

Set $k = 0$.

Step 2. Compute Lie brackets

$$[f, d],\ [g_i, d],\ \forall d(x) \in \mathcal{R}_k(x),$$

and take

$$\mathcal{R}_{k+1}(x) = \mathcal{R}_k(x) + \operatorname{span}\{\text{Lie brackets which are not in } \mathcal{R}_k(x)\}.$$

Step 3. Stop and set $\mathcal{R}_c(x) = \mathcal{R}_{k+1}(x)$ if $\mathcal{R}_{k+1}(x) = \mathcal{R}_k(x)$, or $\dim \mathcal{R}_{k+1}(x) = n, \forall x \in N$. Otherwise, return to Step 2 with $k = k + 1$.

Note: There is no guarantee that the process will end up.

Example

Consider the angular motion of a spacecraft. Here we assume there are only two controls (two pairs of boosters) available. The model for angular velocities around the three principal axes is as follows:

$$\begin{aligned}
\dot{x}_1 &= \frac{a_2 - a_3}{a_1} x_2 x_3 \\
\dot{x}_2 &= \frac{a_3 - a_1}{a_2} x_1 x_3 + u_1 \\
\dot{x}_3 &= \frac{a_1 - a_2}{a_3} x_2 x_1 + u_2 \\
& a_1 > 0,\ a_2 > 0,\ a_3 > 0.
\end{aligned}$$

Let us compute the strong accessibility distribution $R_c(x)$ and check the accessibility of the system. In this case,

$$f(x) := \begin{bmatrix} \alpha x_2 x_3 \\ \beta x_3 x_1 \\ \gamma x_1 x_2 \end{bmatrix},\ g_1(x) := \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ g_2(x) := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

where $\alpha := (a_2 - a_3)/a_1$, $\beta = (a_3 - a_1)/a_2$ and $\gamma = (a_1 - a_2)/a_3$.

Step 1. $\mathcal{R}_0(x) = \text{span}\{g_1(x), g_2(x)\} = \text{span}\{e_2, e_3\}$.

Step 2. Lie brackets are computed as follows:

$$\begin{aligned}
[f, g_1] &= \frac{\partial e_2}{\partial x} f(x) - \frac{\partial f}{\partial x} e_2 = -\begin{bmatrix} \alpha x_3 \\ 0 \\ \gamma x_1 \end{bmatrix} =: g_3(x) \\
[f, g_2] &= \frac{\partial e_3}{\partial x} f(x) - \frac{\partial f}{\partial x} e_3 = -\begin{bmatrix} \alpha x_2 \\ \beta x_1 \\ 0 \end{bmatrix} =: g_4(x) \\
[g_1, g_2] &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Thus,

$$\mathcal{R}_1(x) = \text{span}\{g_i(x), i = 1, \ldots, 4\}.$$

Step 3. If $\alpha = 0$ (i.e. $a_2 = a_3$), then $\mathcal{R}_1(x) = \mathcal{R}_0(x)$. So, $\mathcal{R}_c(x) = \mathcal{R}_0(x) = \text{span}\{e_2, e_3\}$. If $\alpha \neq 0$, then $\mathcal{R}_1(x) \neq \mathcal{R}_0(x)$ and $\dim \mathcal{R}_1(x) = 2 < 3$ for $x_2 = x_3 = 0$. Hence, go back to Step 2.

Step 2-2.

$$\mathcal{R}_2(x) = \mathcal{R}_1(x) + \text{span}\left\{[f, g_i], [g_i, g_j], \ i, j = 1, 2, 3, 4\right\}$$

Since

$$[g_1, g_4] = \frac{\partial g_4}{\partial x} e_2 - \frac{\partial e_2}{\partial x} g_4(x) = \begin{bmatrix} -\alpha \\ 0 \\ 0 \end{bmatrix}, \ (\alpha \neq 0)$$

$\mathcal{R}_2(x) = R^3$ (whole space).

Step 3-2 Since $\dim \mathcal{R}_2(x) = 3$ for any x, $R_c(x) = R^3$.

Therefore, if $a_2 \neq a_3$, then the system is locally strongly accessible from any point in R^3.

2 Stability for linear systems

We will discuss the stability of the linear system

$$\dot{x}(t) = Ax(t).$$

2.1 Asymptotic stability

The matrix A is stable (i.e., all the eigenvalues of A have negative real parts) if and only if for any $N < 0$, there exists a unique solution $P > 0$ for the Lyapunov equation

$$A^T P + PA = N.$$

In this case, if we define the Lyapunov function

$$V(x) := x^T P x,$$

with the solution P, then

$$\dot{V}(x(t)) = \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) = x^T(t)(A^T P + PA) x(t) < 0$$

for $x(t) \neq 0$.

Example

Check if the matrix $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ is stable, without computing the eigenvalues.

Set $N = -I_2$ and solve the Lyapunov equation (you can use `lyap.m`):

$$\underbrace{\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}}_{A^T} \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} + \underbrace{\begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} = - \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{N}$$

$$\Leftrightarrow P = \frac{1}{4} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}.$$

The matrix P is positive definite, and therefore, A is stable.

2.2 Critical stability

Suppose that the matrix A does not have eigenvalues with positive real part and has some eigenvalues on the imaginary axis. Such a case is called a *critical case*. In critical cases, the system is stable if and only if algebraic multiplicities of the eigenvalues on the imaginary axis equal geometric multiplicities.

Example

First, consider the system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x,$$

where A has two eigenvalues at the origin (algebraic multiplicity is two). For the two eigenvalues, there is only one eigenvector $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ (geometric multiplicity is one). Hence, the system is *unstable*. Indeed,

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = 0 \end{cases} \Longrightarrow \begin{cases} x_1(t) = x_{20}t + x_{10} \\ x_2(t) = x_{20} \end{cases}$$

and $x_2(t)$ diverges if $x_{20} \neq 0$.

Next, consider the system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} x,$$

where A has eigenvalues at $\pm i$, with algebraic multiplicity one. Since each eigenvalue corresponds to one eigenvector, algebraic and geometric multiplicities are the same for each eigenvalue. Hence, this system is *stable*. Indeed,

$$\left\{ \begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} x_1(t) = r\sin(t+\phi) \\ x_2(t) = r\cos(t+\phi) \end{array} \right.$$

and the trajectory $\{(x_1(t), x_2(t))\}_t$ forms a circle with radius r.

3 Stability for nonlinear systems

3.1 Principle of stability in the first approximation

Consider a nonlinear system

$$\dot{x} = Ax + g(x),$$

where $g(x)$ indicates higher order terms than order one (i.e., g may include x_1^2, x_1x_2, x_2^2 etc.). Denote the set of all the eigenvalues of A by $\sigma(A)$. Then, $x = 0$ is

- **exponentially stable** if $\sigma(A) \subset C^-$. (C^- is the open left half-plane.)
- **unstable** if $\sigma(A) \cap C^+ \neq \emptyset$. ($C^+$ is the open right half-plane.)

If A has no eigenvalue in the open right half-plane but has at least one eigenvalue on the imaginary axis, then we need nonlinear stability theory, such as center manifold theory, to determine the stability of $x = 0$.

Next, consider a nonlinear system with a control

$$\dot{x} = Ax + g(x) + Bu,$$

where g is the same as above. If (A, B) is stabilizable, then $x = 0$ of the nonlinear system can be exponentially stable by using a state feedback $u = Fx$, where F is chosen so that $A + BF$ is stable.

Example

Consider a nonlinear system

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix}}_{A} x + g(x),$$

where g is a higher order term than order one.

If $\alpha = -1$ **and** $\beta = -2$**:** A has two eigenvalues at -1, and hence $x = 0$ is exponentially stable. (In fact, if α and β are negative, then A is a stable matrix and $x = 0$ is exponentially stable.)

If $\alpha = 0$ **and** $\beta = 1$**:** A has eigenvalues at 1, and hence $x = 0$ is unstable.

If $\alpha = \beta = 0$**:** Since A has eigenvalues only on the imaginary axis, we cannot determine the stability of $x = 0$ by "Principle of stability in the first approximation".

3.2 Stability for a special but important nonlinear system

Consider a scalar nonlinear system

$$\dot{x} = ax^n, \; x(0) = x_0,$$

where a is a real constant and n is a positive integer. Study the stability of this system.

We consider several cases.

If $n = 1$**:** The system is linear.

$$x(t) = e^{at} x_0.$$

If $a < 0$**:** Since $x(t) \to 0$ as $t \to \infty$, $x = 0$ is *asymptotically stable.*
If $a = 0$**:** Since $x(t) = x_0$ for all t, $x = 0$ is *(critically) stable.*
If $a > 0$**:** Since $|x(t)| \to \infty$ as $t \to \infty$, $x = 0$ is *unstable.*

If $n > 1$**:** We solve the differential equation.

$$\begin{aligned} \dot{x} = ax^n \quad & \Rightarrow \quad \int x^{-n} dx = \int a dt \\ & \Rightarrow \quad \frac{1}{1-n} x^{1-n} = at + \frac{1}{1-n} x_0^{1-n}, \; \text{(since } x(0) = x_0\text{)} \\ & \Rightarrow \quad x(t)^{n-1} = \frac{1}{(1-n)at + x_0^{1-n}} \end{aligned}$$

If $a = 0$: Since $x(t) = x_0$ for all t, $x = 0$ is *critically stable.*

If $a \neq 0$: Since $\left|(1-n)at + x_0^{1-n}\right| \to \infty$ as $t \to \infty$, $|x(t)| \to 0$ as $t \to \infty$. **But** the question is if $|x(t)| \to \infty$ at some time $t_0 \in (0, \infty)$ for some x_0.

By setting the denominator of $x(t)^{n-1}$ equal zero,

$$t_0 := \frac{x_0^{1-n}}{a(n-1)} = \frac{1}{ax_0^{n-1}(n-1)}.$$

If $a < 0$ and n is odd: Since $x_0^{n-1} > 0$ for all nonzero x_0, $t_0 < 0$ and hence $|x(t)| \neq \infty$ and $x = 0$ is *asymptotically stable.*

Otherwise: We can always choose x_0 such that

$$t_0 = \frac{1}{ax_0^{n-1}(n-1)} > 0.$$

Thus, $x = 0$ is *unstable.*

In summary, $x = 0$ is

- **asymptotically stable** if $a < 0$ and n is odd,
- **critically stable** if $a = 0$,
- **unstable** otherwise.

Fact

The stability of the system

$$\dot{x} = ax^n + \mathcal{O}(|x|^{n+1})$$

is the same as the stability of $\dot{x} = ax^n$. (This fact will be useful when you learn center manifold theory.)

4 How to check stability in critical cases?

Consider a nonlinear system

$$\dot{x} = f(x).$$

Now suppose that the matrix

$$L := \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

has no eigenvalues in the open right half-plane but has some eigenvalues on the imaginary axis. Such cases are called *critical cases*. To check the stability in such cases, one can use the center manifold theory.

The procedure to check the stability is as follows.

Step 1. From $\dot{x} = f(x)$, obtain

$$\dot{x} = Lx + p(x), \tag{1}$$

where p includes higher order terms than order one.

Step 2. If necessary, do a coordinate change to transform (1) into

$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & \\ & B \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} + \begin{bmatrix} f(z,y) \\ g(z,y) \end{bmatrix},$$

where A and B have eigenvalues only on the imaginary axis and only in the open left half-plane, respectively.

Step 3. First, try to solve $\dot{y} = 0$, i.e.,

$$By + g(z,y) = 0,$$

with respect to y. If it is difficult to solve, we solve instead $By + g(z,0) = 0$, i.e.,

$$y = -B^{-1}g(z,0).$$

Set $\phi(z) := y$. Using the obtained ϕ, define

$$M\phi(z) := \frac{\partial \phi}{\partial z}(Az + f(z,\phi(z))) - B\phi(z) - g(z,\phi(z))$$

Step 4. The center manifold is approximated as

$$h(z) = \phi(z) + \mathcal{O}(M\phi(z)).$$

Step 5. Check the stability of

$$\dot{w} = Aw + f(w, h(w)).$$

Example 1

Check the stability of the system

$$\begin{cases} \dot{x}_1 &= x_1^4 + x_1 x_2 \\ \dot{x}_2 &= -2x_2 - x_1^2 + x_1 x_2^2 \end{cases}$$

This system can be written as

$$\begin{cases} \dot{x}_1 &= \underbrace{0}_{A} x_1 + \underbrace{x_1^4 + x_1 x_2}_{f(x_1,x_2)} \\ \dot{x}_2 &= \underbrace{-2}_{B} x_2 + \underbrace{(-x_1^2) + x_1 x_2^2}_{g(x_1,x_2)}. \end{cases}$$

Since it is difficult to solve $-2x_2 - x_1^2 + x_1 x_2^2 = 0$ with respect to x_2, we set

$$x_2 = -B^{-1} g(x_1, 0) = -\frac{1}{2} x_1^2 =: \phi(x_1).$$

Hence,

$$\begin{aligned} M\phi(x_1) &:= \frac{\partial \phi}{\partial x_1}(Ax_1 + f(x_1, \phi(x_1))) - B\phi(x_1) - g(x_1, \phi(x_1)) \\ &= -x_1 \left(x_1^4 - \frac{1}{2} x_1^3 \right) - \frac{1}{4} x_1^5 \\ &= \mathcal{O}(x_1^4). \end{aligned}$$

So, the center manifold is approximated as

$$h(x_1) = -\frac{1}{2} x_1^2 + \mathcal{O}(x_1^4).$$

Let us check the stability of

$$\begin{aligned} &\dot{w} = \underbrace{0}_{A} w + \underbrace{w^4 + w \left(-\frac{1}{2} w^2 + \mathcal{O}(w^4) \right)}_{f(w, h(w))} \\ &\Rightarrow \dot{w} = -\frac{1}{2} w^3 + \mathcal{O}(w^4). \end{aligned}$$

Since $w = 0$ is asymptotically stable for this system, $(x_1, x_2) = (0, 0)$ is also asymptotically stable for the original system.

Example 2

Consider the control system

$$\begin{cases} \dot{x}_1 &= x_2 x_3 \\ \dot{x}_2 &= u_1 \\ \dot{x}_3 &= -x_1 x_2 + u_2. \end{cases}$$

This is the model of spacecraft with some constants (see the note for Exercise Session 5). If we use control

$$u_1 = -x_2 + x_1^2, \quad u_2 = -x_3 - x_1^3,$$

the closed-loop system becomes

$$\begin{cases} \dot{x}_1 &= x_2 x_3 \\ \dot{x}_2 &= -x_2 + x_1^2 \\ \dot{x}_3 &= -x_1 x_2 - x_3 - x_1^3. \end{cases}$$

We will check the stability of this closed-loop system.

We can write the system as

$$\begin{cases} \dot{x}_1 &= Ax_1 + f(x_1, [x_2, x_3]) \\ \begin{bmatrix} \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= B \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} + g(x_1, [x_2, x_3]) \end{cases}$$

where

$$A = 0,\ B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},\ f(x_1, [x_2, x_3]) = x_2 x_3,\ g(x_1, [x_2, x_3]) = \begin{bmatrix} x_1^2 \\ -x_1 x_2 - x_1^3 \end{bmatrix}.$$

First, solve

$$0 = -x_2 + x_1^2 \Rightarrow x_2 = x_1^2 =: \phi_1(x_1)$$

$$0 = -x_1 x_2 - x_3 - x_1^3 \Rightarrow x_3 = -2x_1^3 =: \phi_2(x_1)$$

Define $\phi(x_1) := \begin{bmatrix} \phi_1(x_1) \\ \phi_2(x_1) \end{bmatrix}$. Then,

$$\begin{aligned} M\phi(x_1) &:= \frac{\partial \phi}{\partial x_1}(Ax_1 + f(x_1, \phi(x_1))) - B\phi(x_1) - g(x_1, \phi(x_1)) \\ &= \begin{bmatrix} 2x_1 \\ -6x_1^2 \end{bmatrix} x_1^2(-2x_1^3) = \begin{bmatrix} \mathcal{O}(x_1^6) \\ \mathcal{O}(x_1^7) \end{bmatrix}. \end{aligned}$$

So, the center manifold is approximated as

$$h(x_1) = \begin{bmatrix} x_1^2 \\ -2x_1^3 \end{bmatrix} + \begin{bmatrix} \mathcal{O}(x_1^6) \\ \mathcal{O}(x_1^7) \end{bmatrix}.$$

Let us check the stability of the system

$$\begin{aligned} &\dot{w} = (w^2 + \mathcal{O}(w^6))(-w^3 + \mathcal{O}(w^7)) \\ &\Rightarrow \dot{w} = -w^5 + \mathcal{O}(w^9). \end{aligned}$$

$w = 0$ of this system is asymptotically stable, and so is $x = 0$ of the original system.