KTH Matematik

Solution to Final Exam of SF2842 Geometric Control Theory

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Allowed written material: All course material (except the old exams and their solutions) and β mathematics handbook.

Solution methods: All conclusions must be properly motivated. Note: the problems are not necessarily ordered in terms of difficulty.

Note! Your personnummer must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades: 31 points give grade C, 37 points B and 43 points give grade A.

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1. Determine if each of the following statements is *true* or *false* and **motivate** (no motivation no score) your answer briefly (for example, to show a statement is false, a counter-example is enough).

(a) Consider a square linear system that is both controllable and observable.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$.

If $V^* \neq 0$, then system (1) can be made unobservable by some $u = Fx$. . .(2p)

Answer: True, by virtue of the definition for V^*.

(b) If the system (1) has relative degree $(r_1, \cdots, r_m)$, then the only reachability (controllability) subspaces the system has are $\{0\}$ and R^n. . . . . (2p) **Answer:** False, since controllability subspaces have nothing to do with outputs.

(c) Consider an exo-system $\dot{w} = Sw, y_r = qw$, where *dimension* $(w) <$ *dimension* (x), all the eigenvalues of S have non-negative real parts and the pair (q, S) is observable. If we define $e = y - qw$ in system (1) and assume $m = 1$, then there are output matrices C (provided A and B are fixed) such that the full information output regulation problem is always solvable, namely solvable for all exo-systems satisfying the assumptions. . . . . (2p)

Answer: True, since it is exactly an output tracking input problem and all solvability conditions are fulfilled .

(d) Consider

$$\dot{x} = \sum_{i=1}^{m} g_i(x) u_i,$$

where $x \in N(0) \subset R^n$ and $m < n$. Suppose $\omega_i(x), i = 1, \cdots, n - m$ are row vectors that are linearly independent on $N(0)$ such that $\omega_i(x) g_j(x) = 0, \forall i, j$.

If for some $i' \leq n-m$, $\frac{\partial z(x)}{\partial x} = \omega_{i'}(x)$, where $z(x)$ is a scalar function, then the system is not controllable in some neighborhood of the origin. .... (2p)

Answer: True. Since $\dot{z}_i(x) = 0$, z_i cannot be controlled.

(e) Consider a nonlinear single-input single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

where $x \in R^n$, $f, g, h \in C^\infty$ and $f(0) = 0,\ h(0) = 0$. If the system has relative degree n around the origin, then the system is exponentially stabilizable by a state feedback. .... (2p)

Answer: True, since the system is exactly linearizable.

2. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} u \\ y &= (1\ 0\ 0\ 0)x. \end{aligned}$$

(a) Find V^*. .... (4p)

Answer: omitted.

(b) Can we find a $u(t)$ such that the corresponding trajectory from $x(0) = 0$ to $x(t_1) = (0\ 0\ 1\ 1)^T$ lies completely in V^*? .... (3p)

Answer: omitted.

(c) Can we find a friend F of V^*, such that $A + BF$ has all the eigenvalues with negative real parts? .... (3p)

Answer: omitted.

3. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 & -1 \\ \alpha_1 & 1 \\ 0 & 0 \\ 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x. \end{aligned}$$

Design real coefficients α_1, α_2 such that the following two conditions are both satisfied simultaneously

(a) the noninteracting control problem is solvable, ... (5p)

Answer: omitted.

(b) the transmission zero(s) is (are all) with a real part of -2. .... (5p)
Answer: omitted.

4. Consider a control system subject to disturbance:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - x_2 + x_3 + u + w_1 \\
\dot{x}_3 &= \alpha x_3 + u \\
y &= x_1,
\end{aligned}$$

where w_1 is an unknown nonzero constant (disturbance) and α is a real constant. *Note that in this problem you are allowed to identify stabilizability with controllability and detectability with observability.*

(a) Is the disturbance decoupling problem (DDP) solvable? .... (2p)
Answer: No. Because $Im[010] \not\subset V^* = \{x_1 = 0, x_2 = 0\}$

(b) When $u = 0$, for what α the state trajectory $x(t)$ is always bounded? .... (3p)
Answer: $\alpha < 0$ since we need the system matrix is Hurwitz.

(c) Show that for almost all values of α the error feedback output regulation problem is solvable if we choose $y_r = 0$ as the reference output (you do not need to design the controller). What is (are) the value(s) of α such that this problem is not solvable? .... (5p)
Answer: $\alpha \neq 0$ gives the observability of ([C -Q],[A P; 0 S]), and $\alpha \neq 1$ makes sure 0 is not a transmission zero of the system. Furthermore (A, B) is controllable for any real α.

5. Consider in a neighborhood N of the origin

$$\begin{aligned}
\dot{x}_1 &= x_2^3 + e^{x_3} u \\
\dot{x}_2 &= -x_2 + \alpha x_1^3 + x_2^2 \\
\dot{x}_3 &= -x_2^3 + e^{x_3} u \\
y &= x_3,
\end{aligned}$$

where α is a constant.

(a) Convert the system into the normal form. .... (3p)
Answer: Let $\xi_1 = x_3$, $z_1 = x_2$, and $x_2 = x_1 - x_3$, then the resulting system reads

$$\begin{aligned}
\dot{\xi}_1 &= -z_1^3 + e^{\xi_1} u \\
\dot{z}_1 &= -z_1 + \alpha(z_2 + \xi_1)^3 + z_1^2 \\
\dot{z}_2 &= 2z_1^3
\end{aligned}$$

$$y = \xi_1$$

(b) Analyze the stability of the zero dynamics in terms of α. .... (3p)

Answer: The zero dynamics is

$$\begin{aligned}\dot{z}_1 &= -z_1 + \alpha z_2^3 + z_1^2 \\ \dot{z}_2 &= 2z_1^3\end{aligned}$$

It is asymptotically stable if $\alpha < 0$, otherwise unstable.

(c) Design a feedback control to stabilize the nonlinear system for the case when the zero dynamics is asymptotically stable. .... (2p)

Answer: $u = e^{-\xi_1}(z_1^3 - \xi_1)$ could be a solution.

(d) Show the system without the output is not exactly linearizable. .... (2p)

Answer: Since the linearized system is not controllable.