KTH Matematik

Solution to Final Exam of SF2842 Geometric Control Theory

March 20 2015

Examiner: Xiaoming Hu, phone 790 71 80, mobile 070-796 78 31.

Allowed written material: All course material (except the old exams, homeworks and their solutions) and β mathematics handbook.

Solution methods: All conclusions must be properly motivated. Note: the problems are not necessarily ordered in terms of difficulty.

Note! Your personnummer must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades: 31 points give grade C, 37 points B and 43 points give grade A.

---

1. Determine if each of the following statements is *true* or *false* and **motivate** (no motivation no score) your answer briefly (for example, to show a statement is false, a counter-example is enough).

(a) Consider a square linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^m$.

If it does not have any (transmission) zero, then it is both controllable and observable. ....(2p)

Solution: True. This can be shown by Hautus test for controllability and observability.

(b) Consider again system (1) and assume B has full column rank. If $R^* = \{0\}$ then the system is invertible. ....(2p)

Solution: True, as is argued in the compendium.

(c) Consider

$$\begin{aligned} \dot{x} &= Ax + Bu + Pw \\ \dot{w} &= Sw \\ y &= Cx, \end{aligned}$$

where u is control and w disturbance. Assume both (A, B) and (A, P) are controllable and (C, A) is observable. If the system that consists of both x and w is not observable, then output regulation with full information is not solvable (here $Q = 0$). ....(2p)

Solution: False. An eigenvalue of S that is tranmission zero of system (A, P, C) is not necessarily transmission zero of system (A, B, C).

(d) Consider a nonlinear single-input single-output system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \qquad (2)$$

where $x \in R^n,\ f, g, h \in C^\infty$ and $f(0) = 0,\ h(0) = 0$.
The following is the linearized approximation of (2):

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx, \end{aligned} \qquad (3)$$

where $A = \frac{\partial f(x)}{\partial x}|_{x=0},\ b = g(0),\ c = \frac{\partial h(x)}{\partial x}|_{x=0}$. Assume system (2) has relative degree r at the origin and is minimum phase. Then the linearized system (3) is also minimum phase. . . . . (2p)

Solution: False, unless the nonlinear zero dynamics is exponentially stable.

(e) Consider again system (2). If the system has relative degree n at the origin, then the system is exponentially stabilizable by a state feedback. . . . . (2p)

Solution: True, since in this case the system is exactly linearizable, thus asymptotically stabilizable as a linear system.

2. Consider the system

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x. \end{aligned}$$

(a) Find V^*. . . . . (3p)

Solution: $V^* = \{x \in R^4, x_1 = x_2 = 0\}$.

(b) Find R^*. . . . . (2p)

Solution: $R^* = \{x \in R^4, x_1 = x_2 = x_3 = 0\}$.

(c) Can we find a friend F of V^*, such that $A + BF$ has all the eigenvalues with negative real parts? . . . . (3p)

Solution: No.

(d) Can we find a friend F of V^* that makes V^* attractive? Namely for any solution $x(t)$ of the closed-loop system $\dot{x} = (A+BF)x$, the Euclidean distance from $x(t)$ to V^* tends to zero as $t \to \infty$. . . . . (2p)

Solution: Yes, if we choose $u_1 = -9x_1 + 4x_2 - x_3$ for example.

3. Consider the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ \alpha & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x,\end{aligned}$$

where α is a real constant.

(a) For what value of α is the noninteracting control problem solvable? ..... (3p)
Solution: $\alpha \neq 1$.

(b) What is the transmission zero(s) of the system when the noninteracting control problem is solvable? .... (2p)
Solution: $s = -1$.

(c) Suppose now the first output y_1 is taken away from the system, namely only y_2 is kept as output. What is the transmission zero(s) of the system now? (5p)
Solution: Case 1. $\alpha = 1$. We can treat $u_1 + u_2$ as one control, then the corresponding SISO system has relative degree 3, which gives $s = 2, 1, -1$. Case 2. $\alpha \neq 1$. Then the only possible zero would be $s = -1$, and this is verified by checking the rank of the system matrix at $s = -1$.

4. Consider:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= Kx + u + q(t),\end{aligned}$$

where $K = (k_1\ k_2\ \cdots\ k_n)$ is **chosen** such that when u and $q(t)$ are set to zero, the system is **exponentially stable**.

(a) Let $q(t) = \alpha t + \beta \sin(\omega t + \phi)$, where $\alpha,\ \beta > 0,\ \omega > 0,\ \phi$ are arbitrary constants. What is the minimum order of the system such that there exists an output $y = cx$ that reconstructs $q(t)$ in stationarity when $u = 0$? ....(3p)
Solution: $n = 4$.

(b) Now let $n = 3$, $y = 4x_1 + x_3$ be the output and $q(t) = \alpha t$ be the disturbance, show that for *almost* all values of $\omega > 0$, the full information output regulation problem with the tracking error $e = y - \cos(\omega t)$ is solvable (*a solution is not required, but you need specify the values of* ω *for which a solution may not exist*). .... (3p)
Solution: The system has zeros at $s = \pm j2$. Thus $\omega = 2$ is the only value for which a solution may not exist.

(c) Let $\omega = 1$. Can we solve the error feedback output regulation problem for the system specified in (b)? (Hint: let $\tilde{x}_3 = x_3 - \{the\ reference\ output\}$. This can reduce the computation considerably). .... (4p)

Solution: We can write down the exo system $\dot{w} = Sw$, where $S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$,
and $q(t) = w_1,\ \cos(t) = w_4$. After the coordinate change, the system is in the form (when setting $u = 0$ and replace x_3 by $\tilde{x}_3$)

$$\begin{aligned} \dot{x} &= Ax + pdw \\ \dot{w} &= Sw \\ e &= 4x_1 + \tilde{x}_3, \end{aligned}$$

where $d = (1\ 0\ 1\ k_3)$. We can verify that (d, S) is observable. Then applying the results in Ch. 6 we can show that the system is observable.

5. Consider in a neighborhood N of the origin

$$\begin{aligned} \dot{x}_1 &= x_1^5 - x_1^2 x_2 + 2u \\ \dot{x}_2 &= -x_2 + \alpha x_1^3 \\ \dot{x}_3 &= x_2^3 + e^{-x_3} u \\ y &= x_1 + 1 - e^{x_3}, \end{aligned}$$

where α is a constant.

(a) Convert the system into the normal form. (Hint: no need to start with one-forms) .... (3p)
Solution: Let $\xi_1 = y = x_1 + 1 - e^{x_3}, z_1 = x_2, z_2 = x_1 + 2(1 - e^{x_3})$, we get

$$\begin{aligned} \dot{z}_1 &= -z_1 + \alpha x_1^3 \\ \dot{z}_2 &= x_1^5 - x_1^2 z_1 - 2e^{x_3} z_1^3 \\ \dot{\xi}_1 &= x_1^5 - x_1^2 z_1 - e^{x_3} z_1^3 + u \\ y &= \xi_1, \end{aligned}$$

where $x_1 = 2\xi_1 - z_2, e^{x_3} = 1 + \xi_1 - z_2$.

(b) Analyze the stability of the zero dynamics in terms of α. .... (3p)
Solution: Zero dynamics:

$$\begin{aligned} \dot{z}_1 &= -z_1 - \alpha z_2^3 \\ \dot{z}_2 &= -z_2^5 - z_2^2 z_1 - 2(1 - z_2) z_1^3. \end{aligned}$$

It is asymptotically stable if $\alpha < 1$, otherwise unstable.

(c) Design a feedback control to stabilize the nonlinear system for the case when the zero dynamics is asymptotically stable. .... (2p)
Solution: omitted.

(d) Show the system without the output is not exactly linearizable. .... (2p)
Solution: Since the linearized system is not controllable.