Exam Maj 23 2013 in SF2852 Optimal Control.

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Allowed books: The formula sheet and β mathematics handbook.

Solution methods: All conclusions should be properly motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Solve the following optimal control problem using PMP

$$\min \int_0^1 u^2(t)dt \quad \text{subject to} \quad \begin{cases} \dot{x} = -x + u \\ x(0) = 1, \ x(1) = 0 \end{cases}$$

   The optimal control as well as the optimal cost should be given. (10p)

2. Consider the scalar linear quadratic optimal control problem

$$\min \int_0^\infty (x^2 + u^2)dt \quad \text{subject to} \quad \dot{x} = -x + u, \ x(0) = 1 \tag{1}$$

   (a) Compute the optimal stabilizing feedback control and the corresponding optimal cost. ....(3p)

   (b) Compute the closed loop poles. .... (1p)

   Now consider the finite truncation of (1)

$$\min \int_0^T (x^2 + u^2)dt \quad \text{subject to} \quad \dot{x} = -x + u, \ x(0) = 1 \tag{2}$$

   (c) Use the Hamilton-Jacobi-Bellman equation to compute the optimal feedback control and the corresponding optimal cost. . (4p)

   (d) Let $p(t, T)$ be the Riccati solution corresponding to (2), where the final time is made explicit as an argument. Compute

$$\lim_{T\to\infty} p(t, T)$$

   and compare with the solution to the ARE corresponding to (1). .... (2p)

3. The path of a moving target is estimated using a radar. At a number of time points $t \in \{1, \ldots, T\}$ measurements are taken. Each measurement results in a set of n_t positions $M_t = \{x_{t1}, x_{t2}, \ldots, x_{tn_t}\}$. One of the positions in M_t is the correct position of the target at that time, whereas the other time points correspond to scatters from other objects. The goal is to determine which set of positions that give the most likely positions of the object.

   The target is moving according to a Gaussian distribution. Given that the position of the target is x_t at time t, the probability distribution of the target in time $t+1$ is given by

$$p(x_{t+1}|x_t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_t - x_{t+1})^2\right).$$

   The likelihood of a path $x_1, x_2, \ldots, x_T$ is then given by

$$L(x_1, \ldots, x_T) = \prod_{t=1}^{T-1} p(x_{t+1}|x_t).$$

   The problem is to find the path $(x_1, \ldots, x_T)$ that has the highest likelihood out of the paths that are consistent with the measurements, i.e., $x_t \in \{x_{t1}, x_{t2}, \ldots, x_{tn_t}\}$ for all $t = 1, \ldots, T$.

   (*a*) Formulate the problem as a dynamic programming problem. (Hint: formulate it in terms of the log-likelihood.) ....(6p)

   (*b*) Use dynamic programming to find the most likely path given the following set of measurements (with T=4).

$$\begin{aligned} M_1 &= \{2, 4\}, \\ M_2 &= \{2, 5, 8\}, \\ M_3 &= \{4, 6, 8\}, \\ M_4 &= \{2, 7, 8\}. \end{aligned}$$

   .... (4p)

4. Consider the optimal control problem

$$\min \int_0^\infty (u - x)^2 dt \quad \text{subject to} \quad \dot{x} = ax + u, \quad x(0) = x_0.$$

   (*a*) Find the optimal stabilizing solution (i.e., $x(t) \to 0$ as $t \to \infty$) for the case $a > -1$. Determine the optimal cost. .... (6p)

   (*b*) What is the optimal control when $a < -1$. .... (3p)

   (*c*) Discuss the optimal control when $a = -1$. ....(1p)

5. Consider the following optimal control problem

$$\max \int_0^T x(t)dt \quad \text{subject to} \quad \begin{cases} \dot{x} = -x^2 + u \\ x(0) = 0 \\ 0 \le u(t) \le 1 \\ \int_0^T u(t)dt = K, \end{cases}$$

where $T > K > 0$ are given.

(a) Introduce the extra state $y = \int u dt$. Define the Hamiltonian and the TPBVP. .... (2p)

(b) Determine the possible optimal controls u. .... (3p)

(c) Draw the phase diagram in (x, λ) and study it to determine the possible switching sequences. .... (3p)

(d) Find $T > K > 0$ such that the control

$$u(t) = \begin{cases} 1 & t < K \\ 0 & t \ge K \end{cases}$$

is *not* optimal. .... (2p)