Examiner: Johan Karlsson, tel. 790 84 40.

Allowed books: The formula sheet and β mathematics handbook, (or Tachenbuch Mathematischer Formeln).

Solution methods: All conclusions should be properly motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Solve the linear quadratic problem

$$\min \int_0^1 (8x(t)^2 + u(t)^2)dt \quad \text{subj. to} \quad \dot{x}(t) = x(t) + u(t), \quad x(0) = x_0$$

Both the optimal feedback control and the optimal cost should be computed.

.... (10p)

2. A consumer receives an amount of resources x_0 to spend during N years. During each year an amount u_k is spent and the remaining resources are adjusted according to

$$x_{k+1} = \theta(x_k - u_k)$$

where $\theta > 0$ is given (due to, e.g., interest rate, part of the resources going bad/old, etc.). The consumer wants to maximize his utility over N years, i.e., he wants to maximize the utility $\sum_{k=0}^{N-1} \sqrt{u_k}$. The resulting optimization problem is

$$\max \sum_{k=0}^{N-1} \sqrt{u_k} \quad \text{subj. to} \quad \begin{cases} x_{k+1} = \theta(x_k - u_k) \\ 0 \le u_k \le x_k, \ x_0 > 0 \text{ is given} \end{cases}$$

(*a*) Formulate the dynamic programming recursion that solves this optimization problem. .... (2p)

(*b*) Show that the optimal cost to go is on the form $J(k, x) = \alpha_k \sqrt{x}$. Derive a recursion for α_k and derive the optimal feedback solution in terms of α_k. .... (6p)

(*c*) Solve the problem when $N = 2$. .... (2p)

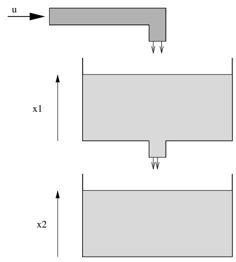

Figure 1: Reservoir system

3. Consider the reservoir system in Figure 1. The system equations are

$$\dot{x}_1(t) = -x_1(t) + u(t)$$
$$\dot{x}_2(t) = x_1(t)$$

where $0 \leq u \leq 1$ and the initial state is $x_1(0) = x_2(0) = 0$. Use PMP to find the control that maximizes $x_2(1)$ subject to the constraint $x_1(1) = 0.5$. .... (10p)

4. Figure 2 illustrates two feedback control laws. The control is equal to $u = 1$ below the switching curve and on the part of the switching curve that is indicated by one of the arrows. Similarly, the control is $u = -1$ above the switching curve and on the part that is indicated by the other arrow. In the left hand figure, all states below the line $x_2 = -x_1 - 1$ and above the line $x_2 = -x_1 + 1$ are uncontrollable in the sense that no control satisfying the constraint $|u| \leq 1$ can drive these states to the origin. Which of the following optimal control problems

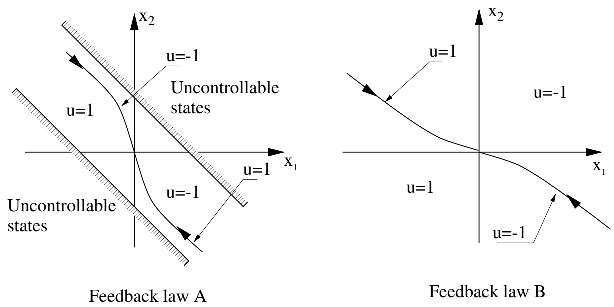

Figure 2: Two feedback control laws.

corresponds to feedback control law A and feedback control law B, respectively? One of the five optimal control problems corresponds to A and one other corresponds to B.

(a)

$$\min T \quad \text{subject to} \quad \begin{cases} \dot{x}_1 = x_2, & x_1(0) = x_{10},\ x_1(T) = 0 \\ \dot{x}_2 = x_1 + u, & x_2(0) = x_{20},\ x_2(T) = 0 \\ |u| \le 1, \quad T \ge 0 \end{cases}$$

(b)

$$\min T \quad \text{subject to} \quad \begin{cases} \dot{x}_1 = x_1, & x_1(0) = x_{10},\ x_1(T) = 0 \\ \dot{x}_2 = x_2 + u, & x_2(0) = x_{20},\ x_2(T) = 0 \\ |u| \le 1, \quad T \ge 0 \end{cases}$$

(c)

$$\min T \quad \text{subject to} \quad \begin{cases} \dot{x}_1 = x_2, & x_1(0) = x_{10},\ x_1(T) = 0 \\ \dot{x}_2 = -x_1 + u, & x_2(0) = x_{20},\ x_2(T) = 0 \\ |u| \le 1, \quad T \ge 0 \end{cases}$$

(d)

$$\min T \quad \text{subject to} \quad \begin{cases} \dot{x}_1 = -3x_1 + 2x_2 + 5u, & x_1(0) = x_{10},\ x_1(T) = 0 \\ \dot{x}_2 = 2x_1 - 3x_2, & x_2(0) = x_{20},\ x_2(T) = 0 \\ |u| \le 1, \quad T \ge 0 \end{cases}$$

(e)

$$\min \int_0^T u(t)^2 dt \quad \text{subject to} \quad \begin{cases} \dot{x}_1 = x_2, & x_1(0) = x_{10},\ x_1(T) = 0 \\ \dot{x}_2 = x_1 + u, & x_2(0) = x_{20},\ x_2(T) = 0 \\ T \text{ is fixed} \end{cases}$$

Hint: You don't need to compute the switching curves. To draw the right conclusions it is enough to determine qualitatively the feedback laws corresponding to the various optimization problems (to do this you can study controllablity of the system matrices, the eigenvalues of the system matrices, etc.). .... (10p)

5. Solve the following infinite horizon control problem

$$\min \int_0^\infty \left(6x(t)^2 + \left(\int_0^t (x(s) + u(s))ds \right)^2 + u(t)^2 \right) dt$$
$$\text{subj. to} \quad \dot{x}(t) = u(t), \quad x(0) = x_0.$$

Give an expression for the optimal "feedback" (describe optimal $u(t)$ in terms of $x(t), x(s)$, and $u(s)$ for $s < t$). Discuss the stability of the resulting controlled system (e.g., closed loop poles). .... (10p)

Solutions

1. The Riccati equation associated with the optimal control problem is

$$\dot{p} + 2p + 8 - p^2 = 0, \quad p(1) = 0$$

By using the separation of variables method we get

$$\frac{dp}{(p+2)(p-4)} = \frac{1}{6}\left(\frac{1}{p-4} - \frac{1}{p+2}\right) dp = dt$$

Integration gives

$$\ln\left(\frac{4-p}{p+2}\right) = 6(t + c_1) \;\Leftrightarrow\; \frac{4-p(t)}{p(t)+2} = c_2 e^{6t}$$

Using the terminal condition $p(1) = 0$ gives $c_2 = 2e^{-6}$ and

$$p(t) = 4\frac{e^{6(1-t)} - 1}{2 + e^{6(1-t)}}$$

Hence, the optimal control is $u(t) = -p(t)x(t)$ and the optimal cost is $J(x_0) = x_0^2 p(0) = x_0^2 4(e^6 - 1)/(e^6 + 2)$.

2. (a) The dynamic programming recursion is given by

$$\begin{aligned} J(k,x) &= \max_{x \ge u \ge 0} \left\{\sqrt{u} + J(k+1, \theta(x-u))\right\}, \quad 0 \le k < N \\ J(N,x) &= 0. \end{aligned}$$

(b) We show this by induction. Clearly $J(N,x) = 0\sqrt{x}$ is on the correct form. We need to show that if $J(k+1,x) = \alpha_{k+1}\sqrt{x}$, then there is a real number α_k such that $J(k,x) = \alpha_k\sqrt{x}$ (also note that the cost is nonnegative, hence $\alpha_k \ge 0$). To see this, plug the assumption into the recursion above.

$$\begin{aligned} J(k,x) &= \max_{x \ge u \ge 0} \left\{\sqrt{u} + J(k+1, \theta(x-u))\right\} \\ &= \max_{x \ge u \ge 0} \left\{\sqrt{u} + \alpha_{k+1}\sqrt{\theta(x-u))}\right\} \\ &= \sqrt{1 + \theta\alpha_{k+1}^2}\sqrt{x}, \end{aligned}$$

which can be seen by analysing the maximum and noting that it is achieved at $u = x/(1+\theta\alpha_{k+1}^2)$ (the optimal feedback). Therefore, $J(k,x) = \alpha_k\sqrt{x}$ and the recursion for α_k is given by

$$\begin{aligned} \alpha_k &= \sqrt{1 + \theta\alpha_{k+1}^2} \\ \alpha_N &= 0. \end{aligned}$$

(c) When $N = 2$, then $\alpha_2 = 0, \alpha_1 = 1, \alpha_0 = \sqrt{1+\theta}$. The feedback is given by $u_0 = x_0/(1+\theta), u_1 = x_1$, and the optimal value is $J(0,x) = \sqrt{(1+\theta)x}$.

3. The optimal control problem has the formulation

$$\max x_2(1) \quad \text{subj. to} \quad \begin{cases} \dot{x}_1(t) = -x_1(t) + u(t), & x_1(0) = 0, x_1(1) = 1/2 \\ \dot{x}_2(t) = x_1(t), & x_2(0) = 0 \\ 0 \leq u \leq 1 \end{cases}$$

The Hamiltonian becomes

$$H(x, u, \lambda) = \lambda_1(-x_1 + u) + \lambda_2 x_1$$

From the pointwise optimization we get

$$u = \text{argmax}_{0 \leq u \leq 1} H(x, u, \lambda) = \begin{cases} 1, & \lambda_1 > 0 \\ 0, & \lambda_1 < 0 \end{cases}$$

We thus expect a switching control law. The adjoint equation becomes

$$\begin{aligned} \dot{\lambda}_1 &= \lambda_1 - \lambda_2 \\ \dot{\lambda}_2 &= 0 \end{aligned}$$

with terminal condition determined by

$$\lambda(1) - \nabla\Phi(x(1)) \perp S_f$$

where $\Phi(x) = x_2$ and $S_f = \{x : x_1(1) = 0.5\}$. Hence, we get $\lambda_1(1) = free$ and $\lambda_2(1) = 1$. We can now solve the adjoint system, which gives

$$\begin{aligned} \lambda_1(t) &= 1 + (\lambda_1(0) - 1)e^t \\ \lambda_2(t) &= 1 \end{aligned}$$

There can be at most one switch in the control function since $\lambda_1(t)$ is a monotonic function. From the problem it is now clear that the control must have the form

$$u(t) = \begin{cases} 1, & 0 \leq t < t_s \\ 0, & t_s < t \leq 1 \end{cases}$$

We can determine the switching time from the constraint $x(1) = 0.5$. We have $x_1(t_s) = 1 - e^{-t_s}$ and

$$x(1) = e^{-(1-t_s)}(1 - e^{-t}) = 0.5$$

which gives $t_s = \ln(\frac{2+e}{2})$.

4. We can immediately exclude the system (e) since the linear quadratic control problem leads to a linear control law. Let us compute the eigenvalues of the system matrices

$$(a)\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{eig} = \pm 1 \qquad (b)\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{eig} = \{1, 1\}$$

$$(c)\quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{eig} = \pm i \qquad (d)\quad \begin{bmatrix} -3 & 2 \\ 2 & -3 \end{bmatrix} \quad \text{eig} = \{-1, -3\}$$

The we can exclude (c) because it has complex eigenvalues, which leads to switching curves that allow for several switches. We can also exclude (b) since the first state is not controllable and the second state only can be controlled to zero inside the region $-1 \leq x_2(t) \leq 1$. System (a) corresponds to the feedback law A. It has one unstable eigenvalue, which implies that not all states can be controled to zero. System (d) thus corresponds to feedback law B.

5. Introduce the state $y(t)$, which is specified by the dynamic $\dot{y} = x + u$ and the initial value $y(0) = 0$. Then the optimization problem can be written as

$$\min \quad \int_0^\infty \left(6x(t)^2 + y(t)^2 + u(t)^2\right) dt$$

$$\text{subject to} \quad \begin{cases} \dot{x} = u, & x(0) = x_0 \\ \dot{y} = x + u, & y(0) = 0. \end{cases}$$

This is an infinite horizon LQ problem with

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; Q = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}, \; R = 1.$$

The optimal feedback is given by $u = -R^{-1}B^T P[x,\, y]^T$ where

$$P = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

is the p.d. solution of ARE.

The closed look system is then

$$\dot{\begin{pmatrix} x \\ y \end{pmatrix}} = (A - BR^{-1}B^T P)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

which is a matrix with eigenvalues in the left half plane ($\lambda = -3/2 \pm \sqrt{5}/2$).