Exam August 16, 2016 in SF2852 Optimal Control.

Examiner: Johan Karlsson, tel. 790 84 40.

Allowed books: The formula sheet and β mathematics handbook, (or Tachenbuch Mathematischer Formeln).

Solution methods: All conclusions should be properly motivated.

Note! Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Solve the optimization problem

$$\min\,(x_3-1)^2+\sum_{k=0}^{2}u_k^2 \quad \text{subj. to} \quad x_{k+1}=x_k+u_k,\ x_0=0$$

using dynamic programming. .... (10p)

2. The following subproblems do not require full solutions. It is enough with an answer and a brief motivation. Remember that the value of a minimization problem is ∞ if the constraint cannot be satisfied.

   (*a*) Consider the optimal control problem

$$\min x_1(T)+x_2(T)+\int_0^T f_0(x,u)dt \quad \text{subject to} \quad \begin{cases}\dot{x}=f(x,u),\\ x(0)=x_0,\\ x_3(T)=1\end{cases}$$

   The state vector has n-variables ($x=\begin{bmatrix}x_1 & x_2 & \dots & x_n\end{bmatrix}^T$). What are the boundary conditions on the adjoint vector λ that can be derived from PMP.

   .... (4p)

   (*b*) Determine the optimal value of the time optimal control problem

$$\min T \quad \text{subj.to} \quad \begin{cases}\dot{x}_1=u, & x_1(0)=1 \quad x_1(T)=0\\ \dot{x}_2=0, & x_2(0)=1, \quad x_2(T)=0\\ |u|\le 1. \end{cases}$$

   .... (2p)

(c) Determine the optimal value of the time optimal control problem

$$\min T \quad \text{subj.to} \quad \begin{cases} \dot{x} = u, & x(0) = 1, \qquad x(T) = 0 \\ |u| \leq 1. \end{cases}$$

.... (2p)

(d) What is the optimal feedback control for the problem

$$\min x(1)^2 + \frac{1}{2}\int_0^1 u^2(t)dt \qquad \begin{cases} \dot{x} = u, & x(0) = 1/2, \\ |u| \leq 1. \end{cases}$$

You may use that the Riccati equation corresponding to the unconstrained problem, i.e., when $|u| \leq 1$ is removed, has the solution

$$p(t) = \frac{1}{3 - 2t}.$$

.... (2p)

3. Consider the infinite horizon optimal control problem

$$\min \int_0^\infty (|x(t)|^p + u(t)^{2m})dt \quad \text{subject to} \quad \begin{cases} \dot{x} = u \\ x(0) = x_0. \end{cases}$$

Here $p \geq 2$ is a given real number and $m \geq 1$ is a given integer.

(a) Compute the optimal feedback and the optimal cost. ...... (8p)

(b) What does the feedback converge to as $m \to \infty$? .... (2p)

4. Consider the problem

$$\min_u \int_0^\infty (y^2 + ru^2)\, dt, \quad \text{subj. to} \quad \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \quad x(0) = x_0 \end{cases}$$

where $r > 0$ is a positive parameter.

(a) Determine the optimal feedback control and the optimal cost. .... (6p)

(b) Plot the root locus, i.e. the closed loop eigenvalue location as a function of the parameter r. .... (4p)

5. Use PMP to solve the optimal control problem

$$\min \int_0^2 (u-1)x dt \quad \text{subject to} \quad \begin{cases} \dot{x} = (2u-1)x, & x(0) = 3,\ x(2) = 2 \\ 0 \leq u \leq 1. \end{cases}$$

Hint: First prove that $x(t) > 0$ when $t \in [0, 2]$. .... (10p)

Solution outline

1. The dynamic programing recursion is

$$V(x, k+1) = \min_u \left\{u^2 + V(x+u, k)\right\}$$
$$V(x, 3) = (x-1)^2$$

   Simple calculations gives

$$u_0 = \frac{1}{4}(1-x_0) = \frac{1}{4}, \quad x_1 = \frac{1}{4}$$
$$u_1 = \frac{1}{3}(1-x_1) = \frac{1}{4}, \quad x_2 = \frac{1}{2}$$
$$u_2 = \frac{1}{2}(1-x_2) = \frac{1}{4}, \quad x_3 = \frac{3}{4}$$

   The optimal cost is given by $V(0, x) = (1-x)^2/4$ and is for $x = 0$ equal to $1/4$.

2. (*a*) $\lambda(T) = \begin{bmatrix} 1 \\ 1 \\ \text{free} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

   (*b*) There is no feasible solution, hence $= \infty$

   (*c*) $J = 1$

   (*d*) The optimal control for the unconstrained problem, $u(t) = -\dfrac{2x(t)}{3-2t}$, satisfies the constraint $|u(t)| \le 1$ and is thus also optimal for the constrained problem. Indeed, the closed loop state satisfies $|x(t)| \le 1$.

3. Infinite time horizon HJBE gives

$$\min_u \left\{|x|^p + u^{2m} + \lambda u\right\} = 0.$$

   By setting the derivative w.r.t. u to zero, we see that the minimizing u is given by

$$u = -\text{sign}(\lambda)\left(\frac{|\lambda|}{2m}\right)^{\frac{1}{2m-1}}.$$

   Next, plug into HJBE

$$|x|^p + \left(\frac{|\lambda|}{2m}\right)^{\frac{2m}{2m-1}} - |\lambda|\left(\frac{|\lambda|}{2m}\right)^{\frac{1}{2m-1}} = 0$$

and solve for λ:

$$\lambda = \text{sign}(x)\alpha|x|^{\beta}$$

where $\alpha = \frac{2m}{(2m-1)^{\frac{2m-1}{2m}}}$ and $\beta = p\frac{2m-1}{2m}$. The optimal cost is hence given by

$$V(x) = \frac{\alpha}{\beta+1}|x|^{\beta+1}$$

and the optimal control

$$u = -\text{sign}(x)\left(\frac{|x|^p}{2m-1}\right)^{\frac{1}{2m}}$$

Noting that $\left(\frac{|x|^p}{2m-1}\right)^{\frac{1}{2m}} \to 1$ as $m \to \infty$ for any $x \neq 0$, the feedback control converges to $u = -\text{sign}(x)$ as $m \to \infty$.

4. (a) The ARE gives the system

$$\begin{aligned} 1 &= \frac{1}{r}P_{12}^2, \\ P_{11} - 10P_{12} &= \frac{1}{r}P_{12}P_{22}, \\ 2P_{12} - 20P_{22} &= \frac{1}{r}P_{22}^2, \end{aligned}$$

with the positive definite solution

$$P = \begin{bmatrix} \sqrt{100r + 2\sqrt{r}} & \sqrt{r} \\ \sqrt{r} & -10r + \sqrt{100r^2 + 2r\sqrt{r}} \end{bmatrix}.$$

and the optimal control

$$\hat{u} = -\frac{1}{\sqrt{r}}x_1 - (\sqrt{100 + \frac{2}{\sqrt{r}}} - 10)x_2.$$

The optimal cost is $J(x_0) = x_0^T P x_0$.

(b) The closed loop system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\sqrt{r}} & -\sqrt{100 + \frac{2}{\sqrt{r}}} \end{bmatrix} x = \hat{A}x.$$

The eigenvalues of $\hat{A}$ have negative real parts, so the closed loop system is stable. The closed loop eigenvalues are located at

$$\lambda = -\sqrt{25 + \frac{1}{2r}} \pm \sqrt{25 - \frac{1}{2r}}$$

If we plot these two eigenvalues in the complex plane as a function of r then we get the root locus. Plot it!

5. We have on $t \in [0,2]$

$$x(t) = e^{\int_0^t (2u(s)-1)ds} > 0.$$

The Hamiltonian becomes $H(x,u,\lambda) = (u-1)x + \lambda(2u-1)x$. Pointwise minimization gives

$$\text{argmin}_{u \in [0,1]}\{(u-1)x + \lambda(2u-1)x\} = \begin{cases} 1, & \sigma < 0 \\ 0, & \sigma > 0 \\ \in [0,1], & \sigma = 0 \end{cases}$$

where the switching function is $\sigma = (2\lambda + 1)$ (since $x(t) > 0$). The adjoint equation is

$$\dot{\lambda} = -(u-1) - (2u-1)\lambda.$$

Hence,

$$\dot{\sigma} = 2\dot{\lambda} = -2(u-1) - 2(2u-1)\lambda$$

Since, $\sigma = 0$ if $\lambda = -1/2$, we get

$$\dot{\sigma}|_{\sigma=0} = 1.$$

This means that we have at most one switch. We have the possible control sequences $\{0\}$, $\{1\}$, and $\{1,0\}$. The first two are impossible since then either $x(2) = 3e^{-2} < 2$ or $x(2) > 2$. We have

$$u(t) = \begin{cases} 1, & 0 \leq t \leq \bar{t} \\ 0, & \bar{t} < t \leq 2 \end{cases}$$

It remains to determine $\bar{t}$. Integration of the system equation gives

$$x(2) = 3e^{-(2-\bar{t})}e^{\bar{t}} = 2.$$

Hence, $\bar{t} = 1 - \frac{1}{2}\ln(3/2)$.