Exam May 31, 2017 in SF2852 Optimal Control.

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Allowed books: The formula sheet and β mathematics handbook, (or Tachenbuch Mathematischer Formeln).

Solution methods: All conclusions should be properly motivated.

Note: Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Consider the optimal control problem

$$\min \int_0^1 (3x(t)^2 + u(t)^2)dt \quad \text{subj. to} \quad \dot{x}(t) = x(t) + u(t), \quad x(0) = x_0$$

   (i) Determine the optimal feedback control. .... (6p)

   (ii) Determine the optimal cost. .... (4p)

2. We will solve two similar optimal control problems.

   (a) Use PMP to solve

$$\min \int_0^2 (u_1(t)^2 + u_2(t)^2)dt \quad \text{subj. to} \quad \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\ x(0) = 0, \ x(2) \in S_2 \end{cases}$$

   where $S_2 = \{x \in R^2 : x_2^2 - x_1 + 1 = 0\}$.
   .... (3p)

   (b) Use PMP to solve

$$\min \int_0^2 (u_1(t)^2 + u_2(t)^2)dt \quad \text{subj. to} \quad \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, \\ x(0) \in S_0, \ x(2) \in S_2 \end{cases}$$

   where $S_0 = \{x \in R^2 : x_2^2 + x_1 = 0\}$ and S_2 is as above. .... (7p)

3. The differential equation

$$\dot{x}(t) = -0.1x(t) + u(t), \quad x(0) = 0$$

describes the reservoir in Figure 1. The variable $x(t)$ corresponds to the hight of the water and $u(t)$ is the net inflow of water to the reservoir at time t. It is assumed that $0 \le u(t) \le M$.

(a) Find the optimal control law that maximizes the cost

$$J = \int_0^{100} x(t)dt$$

.... (3p)

(b) Find the optimal control law that maximizes the cost in (a) subject to the additional control constraint

$$\int_0^{100} u(t)dt = K$$

where K is a given constant that satisfies $0 < K < 100M$. .(7p)

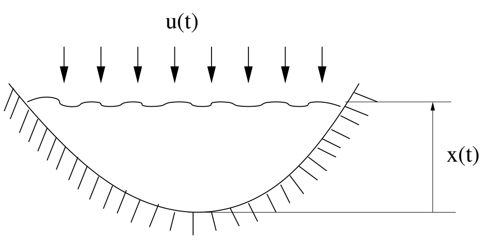

Figure 1: Reservoir.

4. Consider the problem

$$\min_u \int_0^\infty (y^2 + ru^2)\, dt, \quad \text{subj. to} \quad \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x, \quad x(0) = x_0 \end{cases}$$

where $r > 0$ is a positive parameter.

(*a*) Determine the optimal feedback control and the optimal cost. (7p)

(*b*) Determine the closed loop system and compute the closed loop eigenvalue location as a function of the parameter r. ...... (3p)

5. Consider the optimization problem

$$\min T + \frac{1}{2} x_1(T)^2 \quad \text{subj. to} \quad \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ u(t) \end{bmatrix}, \\ x(0) = x_0, \; x_2(T) = 0, \\ |u(t)| \le 1, \\ T \text{ free.} \end{cases}$$

(*a*) Formulate the TPBVP .... (2p)

(*b*) Deduce the structure of any optimal control................ (2p)

(*c*) Assume that the initial condition satisfies $x_2(0) = 0$ and $x_1(0) > 0$. Determine for which values $x_1(0)$ the solution $T^* = 0$ is optimal. .... (3p)

(*d*) Which points in $S_f = \{x \in R^2 : x_2 = 0\}$ may be the final point $x(T)$ for some optimal solution? .... (3p)

Good luck!

1. The Riccati equation (which can easily be derived using PMP or dynamic programming) associated with the optimal control problem is

$$\dot{p} + 2p + 3 - p^2 = 0, \quad p(1) = 0$$

By using the separation of variables method we get

$$\frac{dp}{(p+1)(p-3)} = \frac{1}{4}\left(\frac{1}{p-3} - \frac{1}{p+1}\right) dp = dt$$

Integration gives

$$\ln\left(\frac{p-3}{p+1}\right) = 4(t + c_1) \;\Leftrightarrow\; \frac{p(t)-3}{p(t)+1} = c_2 e^{4t}$$

Using the terminal condition gives $c_2 = -3e^{-4}$ and

$$p(t) = 3\frac{e^{4(1-t)} - 1}{3 + e^{4(1-t)}}$$

(a) $u(t) = -p(t)x(t)$

(b) $J^* = p(0)x_0^2$

2. Both problems have the same solution. Here we only give the proof of part (b), which is a bit harder that (a). The Hamiltonian is

$$H(x, u, \lambda) = u_1^2 + u_2^2 + \lambda_1 u_1 + \lambda_2 u_2$$

Pointwise minimization gives

$$u^* = \mu(\lambda) = \begin{bmatrix} -\lambda_1/2 \\ -\lambda_2/2 \end{bmatrix}$$

The adjoint equation is

$$\begin{bmatrix} \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} \lambda_1^0 \\ \lambda_2^0 \end{bmatrix}$$

The boundary conditions for the adjoint variable reduces to

$$\lambda(0) = \begin{bmatrix} 1 \\ 2_2(0) \end{bmatrix} \nu_1, \qquad \lambda(2) = \begin{bmatrix} -1 \\ 2x_2(2) \end{bmatrix} \nu_2$$

where $\nu_1 \nu_2 \in R$. Since $\lambda(t) = \lambda^0$ (constant) we must have $\nu_2 = -\nu_1$. Clearly, this requires that $x_2(0) = x_2(2) = 0$, which gives the control

$$u = \begin{bmatrix} \nu_1 \\ 0 \end{bmatrix}$$

The solution becomes

$$x(t) = \begin{bmatrix} \nu_1 t \\ 0 \end{bmatrix}$$

In order for $x(2) \in S_1$ we must have $\nu_2 = 0.5$.

3. We only provide a detail solution for (b).

   (a) The optimal solution is $u^*(t) = M$, $t \in [0, 100]$.

   (b) We introduce the state

$$y(t) = \int_0^t u(\tau)d\tau.$$

Then the optimal control problem can be formulated as

$$\min \int_0^{100} -x(t)dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = -0.1x(t) + u(t), & x(0) = 0 \\ \dot{y}(t) = u(t), & y(0) = 0,\ y(100) = K \\ 0 \le u(t) \le M \end{cases}$$

The Hamiltonian is

$$H(x, y, u, \lambda_1, \lambda_2) = -x + \lambda_1(-0.1x + u) + \lambda_2 u$$

Pointwise minimization gives

$$u^* = \begin{cases} M, & \lambda_1 + \lambda_2 < 0 \\ [0, M], & \lambda_1 + \lambda_2 = 0 \\ 0, & \lambda_1 + \lambda_2 > 0 \end{cases}$$

Finally, the adjoint equation becomes

$$\begin{aligned} \dot{\lambda}_1 &= 0.1\lambda_1 + 1, & \lambda_1(100) &= 0 \\ \dot{\lambda}_2 &= 0, & \lambda_2(100) &= \text{free} \end{aligned}$$

Hence $\lambda_2(t) = \lambda_2^0 = const$ and

$$\lambda_1(t) = e^{0.1t}\lambda_1^0 + 10(e^{0.1t} - 1)$$

The boundary constraint $\lambda_1(100) = 0$ gives $\lambda_1^0 = 10(e^{-10} - 1)$ and hence

$$\sigma(t) = \lambda_1(t) + \lambda_2(t) = 10(e^{0.1t-10} - 1) + \lambda_2^0$$

It follows that the switching function is increasing. This implies that we must have the switching sequence $\{M, 0\}$. The switching time is determined by the constraint

$$\int_0^{t_f^*} M dt = Mt_f^* = K \quad \Rightarrow \quad t_f^* = K/M.$$

Hence, the optimal control is

$$u^*(t) = \begin{cases} M, & t \in [0, K/M] \\ 0, & t \in (K/M, 100] \end{cases}$$

4. (a) The ARE gives the system

$$\begin{aligned} 1 &= \frac{1}{r} P_{12}^2, \\ P_{11} - 10P_{12} &= \frac{1}{r} P_{12} P_{22}, \\ 2P_{12} - 20P_{22} &= \frac{1}{r} P_{22}^2, \end{aligned}$$

with the positive definite solution

$$P = \begin{bmatrix} \sqrt{100r + 2\sqrt{r}} & \sqrt{r} \\ \sqrt{r} & -10r + \sqrt{100r^2 + 2r\sqrt{r}} \end{bmatrix}.$$

and the optimal control

$$\hat{u} = -\frac{1}{\sqrt{r}} x_1 - (\sqrt{100 + \frac{2}{\sqrt{r}}} - 10) x_2.$$

The optimal cost is $J(x_0) = x_0^T P x_0$.

(b) The closed loop system is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{\sqrt{r}} & -\sqrt{100 + \frac{2}{\sqrt{r}}} \end{bmatrix} x = \hat{A} x.$$

The eigenvalues of $\hat{A}$ have negative real parts, so the closed loop system is stable. The closed loop eigenvalues are located at

$$\lambda = -\sqrt{25 + \frac{1}{2r}} \pm \sqrt{25 - \frac{1}{2r}}$$

If we plot these two eigenvalues in the complex plane as a function of r.

5. The Hamiltonian is given by

$$H(x, u, \lambda) = 1 + \lambda_1 x_2 + \lambda_2 u,$$

hence the pointwise minimizing u is given by

$$u^* = \begin{cases} 1, & \lambda_2 < 0 \\ [-1, 1], & \lambda_2 = 0 \\ -1, & \lambda_2 > 0 \end{cases}$$

The adjoint equation becomes

$$\begin{aligned}\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0\\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1\end{aligned}$$

,

hence λ_1 is constant and $\lambda_2(t) = \lambda_2(0) - \lambda_1 t$. Next, consider the boundary conditions: $\lambda_2(T)$ is free since $x_2(T)$ is fixed, and $\lambda_1(T) = \frac{\partial \Phi}{\partial x_2}(x(T)) = x_1(T)$. Finally, note that $\lambda_2 \not\equiv 0$, since if $\lambda_2 \equiv 0$ then $\lambda_1 \equiv 0$ which contradicts that $H^*(T) = 0$. Consequently $\lambda_2(t)$ is only zero in at most 1 point and there is at most one switch.

(a) To summarize. The TPBVP is

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -\text{sign}(\lambda_2(t))\\ \dot{\lambda}_1 &= 0\\ \dot{\lambda}_2 & - \lambda_1,\end{aligned}$$

with boundary conditions: $x(0) = x_0,\ x_2(T) = 0, \lambda_1(T) = x_1(T)$.

(b) Since the control is bang-bang with at most one switch, any optimal controller is of the form $[-1, 1]$ or $[1, -1]$.

(c) Assume that the initial condition is $(x_{1,0}, 0)$ where $x_{1,0} > 0$. The optimal control must be of the form

$$u^*(t) = \begin{cases} -1, & t < t_0 \\ 1, & t_0 \le t \le 2t_0 \end{cases} \tag{1}$$

for some $t_0 \ge 0$ (note that there is at most one switch and one needs to return to the x_1-axis). Integrating the dynamics, one arrives at the point $x(2t_0) = (x_{1,0} - t_0^2, 0)^T$ at time $2t_0$, which corresponds to the cost $\frac{1}{2}(x_{1,0} - t_0^2)^2 + 2t_0$. The cost corresponding to $t_0 = 0$ is $\frac{1}{2}x_{1,0}^2$, so the question is if the cost is lower for some $t_0 > 0$. This can only happen if

$$\frac{1}{2}(x_{1,0} - t_0^2)^2 + 2t_0 < \frac{1}{2}x_{1,0}^2$$

$$\Leftrightarrow$$

$$\frac{1}{2}t_0^4 - x_{1,0}t_0^2 + 2t_0 < 0$$

has a solution for some $t_0 > 0$, i.e., if

$$\frac{1}{2}t_0^3 - x_{1,0}t_0 + 2 < 0$$

has a solution for some $t_0 > 0$. The minimum of LHS is at $t_0 = \sqrt{2x_{1,0}/3}$, hence the minimum value of the LHS is

$$(1/3 - 1)\sqrt{2/3}x_{1,0}^{3/2} + 2 = 2 - (2/3x_{1,0})^{3/2},$$

which is greater or equal to zero if and only if $x_{1,0} \leq 3/2^{1/3}$. The solution $T^* = 0$ is only optimal if $x_{1,0} \leq 3/2^{1/3}$.

(*d*) Assume that $x(\hat{T})$ with $x_1(\hat{T}) \geq 0$ is a final point corresponding to an optimal solution, and hence corresponds to the optimal value $\hat{T} + \frac{1}{2}x_1(\hat{T})^2$. Noting that the cost is linear in T, this final point could only correspond to an optimal optimal solution if $x_1(\hat{T}) \leq 3/2^{1/3}$. Otherwise one could use the controller (**??**) according to (c) and achieve a lower cost. By symmetry of the problem the same argument holds for $x_1(\hat{T}) \leq 0$, and hence any optimal solution must satisfy $|x_1(T^*)| \leq 3/2^{1/3}$ and $x_2(T^*) = 0$