Homework 0: SF2852: Optimal Control
Grading: You may hand in the homework and get feedback on your solutions if you hand it in before the exercise session 2.

Problem 1

In the first problem we will compute the dynamics of a linear time-invariant system. Consider the ODE

$$\ddot{y}(t) = 3\dot{y}(t) - 2y(t) + u(t), \quad \text{where} \quad y(0) = 1, \dot{y}(0) = 0.$$

(a) Reduce the system a first order system on the form

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where $x(t) \in \mathbb{R}^{2\times 1}$, $A \in \mathbb{R}^{2\times 2}$, and $B \in \mathbb{R}^{1\times 2}$.
For example, you can let $x(t) = \begin{pmatrix} y(t) & \dot{y}(t) \end{pmatrix}^T$.

(b) Determine the initial condition $x(0)$.

(c) Compute $e^{At} \in \mathbb{R}^{2\times 2}$. Use for example the Laplace transform $e^{At} = \mathcal{L}^{-1}((sI - A)^{-1})(t)$.

(d) Compute the solution $x(t)$ (and $y(t)$) when $u(t) \equiv 1$ by using

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau.$$

Problem 2

The purpose of this problem is to get acquainted with Matlab and the toolbox CVX, which is useful for solving convex optimization problems. The software package can be downloaded from `http://cvxr.com/cvx/`.

Consider a system with the scalar dynamics

$$x_{k+1} = x_k + u_k, \text{ for } k = 0, 1, \dots, T-1,$$

where we want to track a given trajectory y_k but where we also want to use a low control signal. Here $x_k, y_k, u_k \in \mathbf{R}$. This can, e.g., be posed as the following optimization problem

$$\begin{aligned} \min_{u_k, x_k} \quad & |x_T - y_T|^2 + \sum_{k=0}^{T-1} \left(|x_k - y_k|^2 + |u_k|^2 \right) \\ \text{subject to} \quad & x_{k+1} = x_k + u_k, \text{ for } k = 0, 1, \dots, T-1, \\ & x_0 = 0. \end{aligned}$$

An implementation of this is provided below.

- Modify the code so that it solves the output feedback problem

$$\begin{aligned} \min_{u_k, x_k} \quad & |Cx_T - y_T|^2 + \sum_{k=0}^{T-1} \left(|Cx_k - y_k|^2 + |u_k|^2 \right) \\ \text{subject to} \quad & x_{k+1} = Ax_k + Bu_k, \text{ for } k = 0, 1, \dots, T-1, \\ & x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \end{aligned}$$

  where $x_k \in \mathbf{R}^2$ and

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Good luck!

Matlab code

```
% The number of time steps
T=20;

% The time grid
T_grid=(0:T);

% Desired trajectory/Signal
y=sin(0.5*T_grid+pi/4)./(T_grid+2);

cvx_begin
    variable u(1,T)
    variable x(1,T+1)

    % Cost function to be minimized
    minimize( (x-y)*(x-y)' + u*u' )

    % Dynamics of the system
    for k=1:T
        x(k+1) == x(k)+u(k);
    end

    % Initial condition
    x(1)==0

cvx_end;


figure(1)
plot(T_grid, y, T_grid, x, T_grid(1:end-1), u)
legend('Desired trajectory', 'Optimal trajectory', 'Control signal')
```