Exam October 22, 2020 in SF2852 Optimal Control.

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Allowed aids: The formula sheet and mathematics handbook (by Råde and Westergren). (Note that calculator is **not** allowed.)

Solution methods: All conclusions should be properly motivated.

Note: Your personal number must be stated on the cover sheet. Number your pages and write your name on each sheet that you turn in!

Preliminary grades (Credit = exam credit + bonus from homeworks): 23-24 credits give grade Fx (contact examiner asap for further info), 25-27 credits give grade E, 28-32 credits give grade D, 33-38 credits give grade C, 39-44 credits give grade B, and 45 or more credits give grade A.

1. Solve the following optimal control problem using PMP

$$\min \int_0^1 u^2(t)dt \quad \text{subj. to} \quad \begin{cases} \dot{x} = x + u \\ x(0) = 1, \ x(1) = 1 \end{cases}$$

The optimal control as well as the optimal cost should be given. (10p)

2. Let z_k denote the number of university teachers at time k and let y_k denote the number of scientists at time k. The number of teachers and scientists evolve according to the equations

$$\begin{aligned} z_{k+1} &= (1-\delta)z_k + \gamma z_k u_k, \\ y_{k+1} &= (1-\delta)y_k + \gamma z_k(1-u_k) \end{aligned}$$

where $0 < \delta < 1$ and $\gamma > 0$ are constants and $0 < \alpha \leq u_k \leq \beta < 1$. The interpretation of these equations is that by controlling the funding of the university system it is possible to control the fraction of newly educated teachers that become scientists, i.e. funding affects the control u_k. We assume $z_0 > 0$ and $y_0 = 0$ and in the above equations we allow both z_k and y_k to be non-integer valued in order to simplify the problem. This is a reasonable approximation if, for example, one unit is 10^5 persons.

(*a*) We would like to determine the control u_k so that the number of scientists is maximal at year N. Formulate this as a sequential optimization problem and state the dynamic programming recursion that solves this problem. .... (5p)

(*b*) Use dynamic programming to solve the problem in (*a*) when $\delta = 0.5$, $\gamma = 0.5$, $\alpha = 0.3$, $\beta = 0.8$, $z_0 = 1$, $y_0 = 0$ and $N = 2$. In particular, give the optimal control sequence and the optimal cost. .... (5p)

3. Determine the optimal stabilizing state feedback control corresponding to the following optimal control problem

$$\min \int_0^\infty \frac{1}{2}(x^2 + u^2)dt \quad \text{subj. to} \quad \dot{x} = x + x^2 + u, \quad x(0) = x_0$$

.... (10p)

4. Consider the following infinite horizon optimal control problem

$$\min \int_0^\infty \left((x_1(t) + x_2(t))^2 + u(t)^2\right) dt \quad \text{s.t.} \quad \begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u(t), \\ x_1(0) = x_{10}, \\ x_2(0) = x_{20}. \end{cases} \tag{1}$$

   (*a*) Formulate the problem on the standard form

$$\min \int_0^\infty \left(x(t)^T Q x(t) + u(t)^T R u(t)\right) dt \text{ subj. to } \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0, \end{cases}$$

   i.e. provide the values for all matrices and vectors. .... (1p)

   (*b*) Do the factorization $Q = C^T C$ and verify that (C, A) is observable and (A, B) is controllable, i.e. verify that the following matrices

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \mathcal{C} = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$$

   have full rank (n denotes the dimension of system in the optimal control problem).
   .... (2p)

   (*c*) Determine the optimal stabilizing state feedback solution to the optimal control problem (1). .... (5p)

   (*d*) Is the solution in (*c*) unique? .... (1p)

   (*e*) Verify that the closed loop system is stable? .... (1p)

5. Use PMP to solve the following optimal control problem

$$\min t_f + \int_0^{t_f} |u(t)| dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = -ax(t) + u(t) \\ x(0) = x_0, \; x(t_f) = 0 \\ t_f \geq 0, \; |u| \leq 1 \end{cases}$$

   where $a > 0$ and $x_0 \neq 0$.

.... (10p)

Good luck!

Solutions

1. The Hamiltonian is given by $H(x, u, \lambda) = u^2 + \lambda(x + u)$. Pointwise minimization gives $\tilde{\mu}(x, \lambda) = -\lambda/2$. The adjoint system is

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -\lambda,$$

hence the solution is on the form $\lambda(t) = \lambda_0 \exp(-t)$. The optimal control is on the form $u(t) = -\lambda_0 \exp(-t)/2$. Since there is no constraint on $\lambda(1)$, we need to determine the constant λ_0.

The system satisfies $\dot{x} = x - \lambda_0 \exp(-t)/2$, thus $x(1)$ is given by

$$\begin{aligned} 1 = x(1) &= ex(0) + \int_0^1 e^{1-s} u(s) ds \\ &= e - \frac{\lambda_0}{2} \int_0^1 e^{1-2s} ds \\ &= e - \frac{\lambda_0}{2} [-e^{1-2s}/2]_0^1 \\ &= e + \frac{\lambda_0}{4}(e - e^{-1}), \end{aligned}$$

thus $\lambda_0 = \frac{4(1-e)}{e^{-1}-e}$. The optimal control is thus

$$u(t) = -2\frac{1-e}{e^{-1}-e} e^{-t},$$

and the corresponding cost is $\lambda_0^2(1 - e^{-2})/8$.

2. (a) Let

$$x_k = \begin{bmatrix} z_k \\ y_k \end{bmatrix}, \quad x_0 = \begin{bmatrix} z_0 \\ 0 \end{bmatrix}, \quad f(x, u) = \begin{bmatrix} (1-\delta)z + \gamma z u, \\ (1-\delta)y + \gamma z(1-u) \end{bmatrix}$$
$$U = [\alpha, \beta]$$

With this notation the optimal control problem can be formulated as

$$\max \begin{bmatrix} 0 & 1 \end{bmatrix} x_N \quad \text{subj. to} \quad \begin{cases} x_{k+1} = f(x_k, u_k), & x_0 \text{ given} \\ u_k \in U \end{cases}$$

The dynamic programming recursion that solves this problem can be formulated as

$$\begin{aligned} V(k, x) &= \min_{u \in U} V(k+1, f(x, u)) \\ V(N, x) &= y_N = \begin{bmatrix} 0 & 1 \end{bmatrix} x_N \end{aligned}$$

(b) Note that by the above choise of parameters we always have $x_k > 0$, $k \geq 0$ (component-wise). The dynamic programming recursion gives

$$
\begin{aligned}
V(2, x) &= y \\
V(1, x) &= \max_{0.3 \leq u \leq 0.8} 0.5y + 0.5z(1-u) = 0.5y + 0.35z, \ u_1^* = 0.3
\end{aligned}
$$

and finally

$$
\begin{aligned}
V(0, x) &= \max_{0.3 \leq u \leq 0.8} 0.5(0.5y + 0.5z(1-u)) + 0.35(0.5z + 0.5zu) \\
&= \max_{0.3 \leq u \leq 0.8} \frac{1}{4}y + \frac{17}{40}z - \frac{3}{40}zu \\
&= 0.25y + 0.4025z, \ u_0^* = 0.3
\end{aligned}
$$

We get the following optimal solution $u_0^* = 0.3$ and

$$
\begin{aligned}
z_1^* &= 0.5 \cdot 1 + 0.5 \cdot 1 \cdot 0.3 = 0.65 \\
y_1^* &= 0.5 \cdot 0 + 0.5 \cdot 1 \cdot 0.7 = 0.35
\end{aligned}
$$

and $u_1^* = 0.3$ and

$$
\begin{aligned}
z_2 &= 0.5 \cdot 0.65 + 0.5 \cdot 0.65 \cdot 0.3 = 0.4225 \\
y_2 &= 0.5 \cdot 0.35 + 0.5 \cdot 0.65 \cdot 0.7 = 0.4025
\end{aligned}
$$

3. We need to find a positive definite radially unbounded C^1 function that satisfies the HJBE

$$
0 = \min_u \{\frac{1}{2}(x^2 + u^2) + V'(x)(x + x^2 + u)\} = \frac{x^2}{2} - \frac{1}{2}(V'(x))^2 + V'(x)(x + x^2)\}
$$

The HJBE is an ordinary differential equation in this scalar case. We can easily see that

$$
\begin{aligned}
V'(x) &= x + x^2 \pm \sqrt{(x + x^2)^2 + x^2} \\
&= x + x^2 + x\sqrt{(1 + x)^2 + 1}
\end{aligned}
$$

The second equality follows since this case is necessary to obtain a positive definite solution. The optimal feedback control is

$$
u^*(x) = -(x + x^2 + x\sqrt{1 + (1 + x)^2})
$$

4. (a) The optimal control problem is defined in terms of the following matrices

$$
Q = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \ R = 1, \ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}
$$

(*b*) We have $Q = C^TC$ with $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$. The observability and controllability matrices are

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which both are invertible and thus of full rank. Hence, (C, A) is observable and (B, A) is controllable.

(*c*) The algebraic Riccati equation becomes

$$\begin{aligned} &A^TP + PA + Q - PBR^{-1}B^TP \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix} \begin{bmatrix} p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

This gives the equation system

$$\begin{cases} -p_{12}^2 + 1 = 0 \\ -p_{22}^2 + 2p_{12} + 1 = 0 \\ p_{11} - p_{12}p_{22} + 1 = 0. \end{cases}$$

First note that $p_{12} = \pm 1$. Then note that $p_{22}^2 = 1 + 2p_{12}$, which is negative unless $p_{12} = 1$, thus we must have that $p_{22} = \sqrt{3}$. Finally $p_{11} = p_{12}p_{22} - 1 = \sqrt{3} - 1$, and the unique positive definite solution to the ARE is

$$P = \begin{bmatrix} \sqrt{3} - 1 & 1 \\ 1 & \sqrt{3} \end{bmatrix}$$

which implies that the stabilizing state feedback is

$$u^* = -R^{-1}B^TPx = -\begin{bmatrix} 1 & \sqrt{3} \end{bmatrix} x = -x_1 - \sqrt{3}x_2$$

(*d*) The postive definite solution to the ARE is unique since (C, A) is observable and (A, B) is controllable.

(*e*) The closed loop system is

$$\dot{x} = (A - BR^{-1}B^TP)x = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{3} \end{bmatrix} x$$

The system matrix has both eigenvalues in the left half plane and is thus stable.

5. Since $t_f = \int_0^{t_f} 1 \cdot dt$ it follows that the Hamiltonian is

$$H(x, u, \lambda) = 1 + |u| + \lambda(-ax + u).$$

Pointwise minimization gives

$$\mu(x,\lambda) = \mathrm{argmin}_{|u|\leq 1} H(x,u,\lambda) = \begin{cases} 1, & \lambda < -1 \\ [0,1], & \lambda = -1 \\ 0, & -1 < \lambda < 1 \\ [-1,0], & \lambda = 1 \\ -1, & \lambda > 1 \end{cases}$$

In this problem the adjoint equation becomes

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = a\lambda$$

which has the general solution $\lambda(t) = e^{at}\lambda_i$, where λ_i is a constant. First notice that the case $\lambda_i = 0$ is impossible because then $u^* \equiv 0$, which is impossible because the constraint $x(t_f) = 0$ would be violated. We have the following possible switching sequences for the control

$$\{1\}, \quad \{0,1\}, \quad \{0,-1\}, \quad \{-1\}$$

Consider, for example, the switching sequence $\{0,-1\}$. At the switching instant t_s we have $\lambda(t_s) = 1$ and $u^*(t_s) \in [-1,0]$, which implies $|u^*(t_s)| = -u^*(t_s)$. If we use this in the second PMP condition $H(x^*(t), u^*(t), \lambda(t)) = 0$, $t \in [0, t_f]$ we get the equation

$$H(x^*(t_s), u^*(t_s), \lambda(t_s)) = 1 - u^*(t_s) - ax^*(t_s) + u^*(t_s) = 0,$$

which implies that $x^*(t_s) = 1/a$. If $x_0 > 1/a$ we get the following solution to the differential equation

$$x(t) = \begin{cases} e^{-at}x_0, & x(t) > 1/a \\ e^{-a(t-t_s)}\frac{1}{a} - (1 - e^{-a(t-t_s)})\frac{1}{a}, & 0 < x(t) \leq 1/a \end{cases}$$

Notice that we always have

$$H(x^*(t), u^*(t), \lambda(t)) \equiv 0$$

so the second PMP condition is satisfied. The situation is by symmetry analogous for the switching sequence $\{0,1\}$. Hence, we have derived the optimal feedback control law

$$u^*(t) = \begin{cases} 0, & x(t) > 1/a \\ -1, & 0 < x(t) \leq 1/a \\ 0, & x(t) = 0 \\ 1, & -1/a \leq x(t) < 0 \\ 0, & x(t) < -1/a. \end{cases}$$

We may compute the switching time as follows

$$|x(t_s)| = e^{-at_s}|x_0| = 1/a$$

which gives $t_s = \ln(a|x_0|)/a$. Finally, the optimal terminal time is computed as

$$x(t_f) = e^{-a(t_f - t_s)}x(t_s) + \frac{1}{a}(1 - e^{-a(t_f - t_s)})\text{sign}(x_0) = \frac{1}{a}(2 - e^{-a(t_f - t_s)})\text{sign}(x_0) = 0$$

which gives

$$t_f = t_s - \ln(2)/a = (\ln(ax_0) - \ln(2))/a.$$