Transient

State i is transient if there is a non-zero probability that we will never return to state i.

Recurrent

If a state i is not transient, then it is said to be recurrent.

Absorbing

A state i is called absorbing if it is impossible to leave this state, i.e.,

$$P_{ii} = 1 \quad \text{and} \quad P_{ij} = 0 \text{ for } i \neq j$$

Irreducible

A Markov chain is irreducible if it is possible to get to any state from any state.

Periodic

A state i is periodic with period k if any return to state i must occur in multiples of k time steps, where $k > 1$.

If $k = 1$, the state i is said to be aperiodic (returns to state i can occur at irregular times)

Exercise

A machine manufactures a certain type of components. The components are classified as excellent, acceptable or faulty.

If a component is excellent, the probability of the next component to be excellent is 0,9 and acceptable is 0,1.

If a component is acceptable, the probability of the next component to be excellent is 0,1, acceptable 0,7 and faulty is 0,2.

If a component is faulty the machine is repaired and the next component will be excellent.

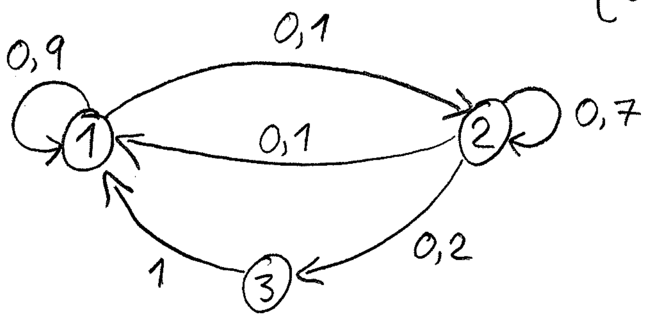
a) Draw a graph of this Markov chain.

b) Construct the transition matrix of this Markov chain.

c) Determine all stationary distributions of the Markov chain.

Exercise

a) Introduce the states $\begin{cases} 1. \text{ excellent} \\ 2. \text{ acceptable} \\ 3. \text{ faulty} \end{cases}$



b) Transition matrix

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} = \begin{pmatrix} 0,9 & 0,1 & 0 \\ 0,1 & 0,7 & 0,2 \\ 1 & 0 & 0 \end{pmatrix}$$

(note that each row sum to one)

c) The stationary probability distribution is determined by

$$\pi P = \pi \quad \text{and} \quad \sum_i \pi_i = 1$$

$$(0,9\pi_1 + 0,1\pi_2 + \pi_3,\ 0,1\pi_1 + 0,7\pi_2,\ 0,2\pi_2) = (\pi_1, \pi_2, \pi_3)$$

gives

$$\begin{cases} -0,1\pi_1 + 0,1\pi_2 + \pi_3 = 0 \\ 0,1\pi_1 - 0,3\pi_2 = 0 & \Rightarrow \pi_1 = \frac{0,3}{0,1}\pi_2 = 3\pi_2 \\ 0,2\pi_2 - \pi_3 = 0 & \Rightarrow \pi_3 = 0,2\pi_2 \end{cases}$$

$$\Rightarrow \pi_1 + \pi_2 + \pi_3 = 3\pi_2 + \pi_2 + 0,2\pi_2 = \frac{21}{5}\pi_2 = 1$$

$$\Rightarrow \pi_2 = \frac{5}{21}$$

$$\Rightarrow \pi_1 = \frac{5}{7}, \quad \pi_3 = \frac{1}{21}$$

So the stationary probability distribution to P is

$$\pi = \left(\frac{5}{7}, \frac{5}{21}, \frac{1}{21}\right)$$

Markov chains in continuous time

Ex

A Markov process $X(t)$, $t \geq 0$ has the states $\{0, 1, 2\}$ and the following intensity matrix

$$Q = \begin{pmatrix} -8 & \square & 4 \\ \square & -5 & 2 \\ 0 & 2 & \square \end{pmatrix}$$

a) Fill in the blanks in the matrix

Row sums are equal to zero, as

$$\left\{ q_{ij} = P_{ij}'(0) \quad \text{and} \quad \sum_j P_{ij}(t) = 1 \Rightarrow \sum_j q_{ij} = \sum_j P_{ij}'(0) = 0 \right\}$$

$$Q = \begin{pmatrix} -8 & 4 & 4 \\ 3 & -5 & 2 \\ 0 & 2 & -2 \end{pmatrix}$$

b) Determine all stationary distributions π of the Markov. process.

$$\pi = \pi P(t), \ t \geq 0 \qquad \text{and} \quad \sum \pi_j = 1$$

derivative

$$0 = \pi Q \qquad \text{and} \quad \sum_j \pi_j = 1$$

(ok if finite number of states).

$$\pi Q = (-8\pi_1 + 3\pi_2, \ 4\pi_1 - 5\pi_2 + 2\pi_3, \ 4\pi_1 + 2\pi_2 - 2\pi_3) = 0$$

$\Rightarrow \pi_1 = \frac{3}{8}\pi_2$ and $\pi_1 + \pi_2 + \pi_3 = 1$ gives $\pi_3 = 1 - (1 + \frac{3}{8})\pi_2$

we have

$\pi = (\frac{3}{8}\pi_2, \ \pi_2, \ 1 - \frac{11}{8}\pi_2)$ together with $\pi_i \geq 0 \ \forall i$

$\Rightarrow \pi = (\frac{3}{8}t, \ t, \ 1 - \frac{11}{8}t)$ where $t \in [0, \frac{8}{11}]$.