Tutorial 5: Inventory Theory

2.5 Given:

Ingoing inventory x

Price per unit c kr

Inventory cost h kr / unit ° time unit

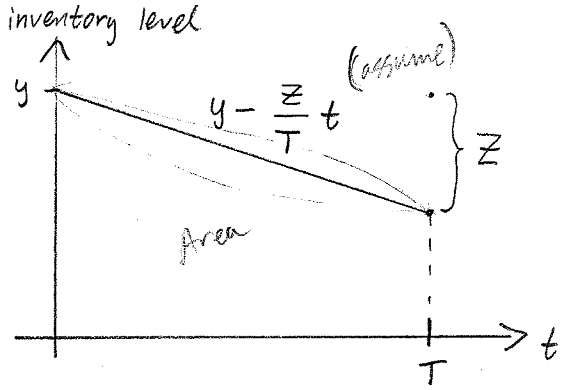
Shortage cost p kr / unit ° time unit

The demand during the period is a random variable Z with probability density function f.

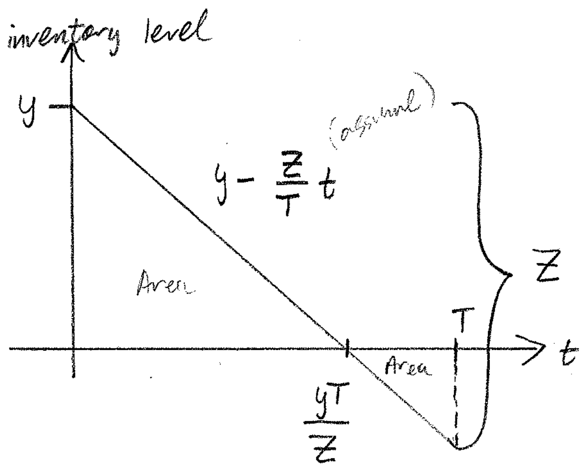
We seek the optimal inventory level, assume $y \geq x$.

Assume that the demand per time unit is constant over the time interval T.

There are two possible scenarios:



$Z \leq y$



$Z > y$

The total cost is $(y > x)$

$$C(y, Z) = \begin{cases} c(y-x) + h\int_0^T \left(y - \frac{Z}{T}t\right) dt, & Z \leq y \\ c(y-x) + h\int_0^{yT/Z} \left(y - \frac{Z}{T}t\right) dt + p\int_{yT/Z}^{T} (-1)\left(y - \frac{Z}{T}t\right) dt, & Z > y \end{cases}$$

$$C(y, Z) = \begin{cases} c(y-x) + h\left((y-Z)T + \frac{ZT}{2}\right), & Z \leq y \\ c(y-x) + h\cdot\frac{y^2T}{2Z} + p\cdot\underbrace{\left(-yT + \frac{y^2T}{Z} + Z\frac{T}{2} - \frac{\cancel{Z}}{\cancel{T}}\frac{y^2T^{\cancel{2}}}{2Z^{\cancel{2}}}\right)}_{\left(\frac{ZT}{2} - yT + \frac{y^2T}{2Z}\right)}, & Z > y \end{cases}$$

Compute the expected value:

$$C(y) = c(y-x) + h\int_0^y \left(yT - \frac{zT}{2} \right) f(z)\, dz \; + $$
$$+ h \int_y^\infty \frac{y^2 T}{2z} f(z)\, dz + \frac{p}{2} \int_y^\infty \left(zT - 2yT + \frac{y^2 T}{z} \right) f(z)\, dz$$

Want to minimize C w.r.t. y.
How to compute $C'(y)$?

$$\boxed{\begin{aligned} \frac{\partial}{\partial y} \int_{u(y)}^{v(y)} g(y,z)\, dz &= g(y, v(y)) \frac{d v(y)}{dy} - g(y, u(v)) \frac{du(y)}{dy} \; + \\ &+ \int_{u(y)}^{v(y)} \frac{\partial}{\partial y} g(y,z)\, dz \end{aligned}}$$

$$C'(y) = c + \cancel{h \cdot \frac{yT}{2} f(y) \cdot 1} - 0 + h \cdot \int_0^y T f(z)\, dz \; +$$
$$+ 0 - \cancel{h \cdot \frac{yT}{2} \cdot f(y) \cdot 1} + h \int_y^\infty \frac{yT}{z} f(z)\, dz$$
$$+ 0 - \frac{p}{2} \underbrace{\left(yT - 2yT + \frac{y^{\cancel{2}} T}{\cancel{y}} \right)}_{=0} f(y) + \frac{p}{\cancel{2}} \int_y^\infty \left(-\cancel{2}T + \frac{\cancel{2} yT}{z} \right) f(z)\, dz$$

$$C'(y) = c + h \cdot \int_0^y T f(z)\, dz + h \cdot \int_y^\infty \frac{yT}{z} f(z)\, dz \; +$$
$$+ p \int_y^\infty \left(\frac{yT}{z} - T \right) f(z)\, dz$$

$$C''(y) = \cancel{hTf(y)}\cdot 1 - 0 + 0 +$$
$$+0 - \cancel{hTf(y)}\cdot 1 + h\int_y^\infty \frac{T}{z} f(z)\,dz +$$
$$+0 - p\underbrace{(T-T)}_{=0} f(y)\cdot 1 + p\int_y^\infty \frac{T}{z} f(z)\,dz$$
$$= T(h+p)\underbrace{\int_y^\infty \frac{1}{z} f(z)\,dz}_{>0,\ \text{since } \int_y^\infty f(z)\,dz = P(Z>y)>0.} > 0$$

Thus C is convex $\Rightarrow$ the minimum y_0 is given by $C'(y_0) = 0$.

$$C'(y_0) = c + hT\int_0^{y_0} f(z)\,dz + (h+p)Ty_0\int_{y_0}^\infty \frac{1}{z} f(z)\,dz - pT\underbrace{\int_{y_0}^\infty f(z)\,dz}_{= 1 - \int_0^{y_0} f(z)\,dz,}$$

since $\int_0^\infty f(z)\,dz = 1$

$$C'(y_0) = c + T(h+p)\int_0^{y_0} f(z)\,dz + y_0 T(h+p)\int_{y_0}^\infty \frac{1}{z} f(z)\,dz - pT = 0$$

$$\Rightarrow \int_0^{y_0} f(z)\,dz + y_0\int_{y_0}^\infty \frac{1}{z} f(z)\,dz = \frac{p - c/T}{h+p} \qquad (*)$$

The optimal y is given by

$$\hat{y} = \begin{cases} y_0, & \text{if } y_0 > x \\ x, & \text{if } y_0 \le x \end{cases}$$

b) Let $T = 1$

$c = 0{,}14$

$h = 0{,}5$ and $f(z) = \begin{cases} kz, & 0 \le z \le 10 \\ 0, & \text{otherwise} \end{cases}$

$p = 0{,}5$

$x = 3$

Start by determining the constant k:

$$\int_{-\infty}^{\infty} f(z)\,dz = 1 \Rightarrow 1 = \int_0^{10} kz\,dz = \left[k\frac{z^2}{2}\right]_0^{10} = 50k$$

$$\Rightarrow k = 1/50$$

Put this info $(*)$, gives

$$\int_0^{y_0} \frac{1}{50} z + y_0 \int_{y_0}^{10} \frac{1}{z} \cdot \frac{1}{50} z \, dz = \frac{p - c/T}{p + h}$$

$$\frac{y_0^2}{100} + \frac{y_0}{50}(10 - y_0) = 0{,}36$$

$$y_0^2 - 20y_0 + 36 = 0$$

$$y_0 = 10 \overset{(+)}{-} 8 = 2$$ ($y_0 = 18$ is not possible since $0 \le y_0 \le 10$).

$$x = 3 \Rightarrow y_0 < x \Rightarrow \hat{y} = x = 3.$$