Applying Marginal Allocation

$$C(S) = cS + h\sum_{d=0}^{S-1}(S-d)P_D(d) + \tilde{p}\sum_{d=S}^{\infty}(d-S)P_D(d)$$

Note: $$c\sum_{d=0}^{S-1}(S-d)P_D(d) - c\sum_{d=S}^{\infty}(d-S)P_D(d) =$$

$$= cS\sum_{d=0}^{\infty}P_D(d) - c\sum_{d=0}^{\infty}dP_D(d) = cS - cE[D].$$

$$\Rightarrow \quad C(S) = \underbrace{(c+h)}_{\substack{=\alpha \\ =C_{over}}}\underbrace{\sum_{d=0}^{S-1}(S-d)P_D(d)}_{=g(S)} + \underbrace{(\tilde{p}-c)}_{\substack{=\beta \\ =C_{under}}}\underbrace{\sum_{d=S}^{\infty}(d-S)P_D(d)}_{=f(S)} + \underbrace{cE[D]}_{\text{independent of } S}$$

$$C(S) = \alpha g(S) + \beta f(S) + \text{constant.}$$

Note:

$$\Delta g(S) = g(S+1) - g(S) = \sum_{d=0}^{S}(S+1-d)P_D(d) - \sum_{d=0}^{S-1}(S-d)P_D(d)$$

$$= \sum_{d=0}^{S}P_D(d) = P(D \leq S) = F_D(S) \geq 0 \quad \Rightarrow g \text{ increasing}$$

$$\Delta^2 g(S) = \Delta g(S+1) - \Delta g(S) = \sum_{d=0}^{S+1}P_D(d) - \sum_{d=0}^{S}P_D(d) = P_D(S+1) \geq 0$$

$$\Rightarrow g \text{ integer-convex}$$

$$\Delta f(S) = f(S+1) - f(S) = \sum_{d=S+1}^{\infty}(d-S-1)P_D(d) - \sum_{d=S}^{\infty}(d-S)P_D(d)$$

$$= -\sum_{d=S+1}^{\infty}P_D(d) = -P(D \geq S+1) = -(1-F_D(S)) \leq 0$$

$$\Rightarrow f \text{ decreasing}$$

$$\Delta^2 f(S) = \Delta f(S+1) - \Delta f(S) = -\sum_{d=S+2}^{\infty}P_D(d) + \sum_{d=S+1}^{\infty}P_D(d) = P_D(S+1) \geq 0$$

$$\Rightarrow f \text{ integer-convex}$$

Prop 3.1: $\hat{x}$ minimizes $\alpha g(x) + \beta f(x)$ iff

$$-\frac{\Delta f(\hat{x})}{\Delta g(\hat{x})} \leq \frac{\alpha}{\beta} \leq -\frac{\Delta f(\hat{x}-1)}{\Delta g(\hat{x}-1)} \text{ if } \hat{x}>0, \qquad -\frac{\Delta f(0)}{\Delta g(0)} \leq \frac{\alpha}{\beta} \text{ if } \hat{x}=0.$$

Here: $\Delta f(S) = -[1-F_D(S)] \qquad \Delta g(S) = F_D(S)$

$$\frac{1-F_D(S)}{F_D(S)} \leq \frac{\alpha}{\beta} \leq \frac{1-F_D(S-1)}{F_D(S-1)}, \qquad \frac{1-F_D(0)}{F_D(0)} \leq \frac{\alpha}{\beta}$$

$$\frac{1-F}{F} \leq \frac{\alpha}{\beta} \Leftrightarrow 1-F \leq \frac{\alpha}{\beta}F \Leftrightarrow 1 \leq \left(1+\frac{\alpha}{\beta}\right)F \Leftrightarrow \frac{\beta}{\alpha+\beta} \leq F$$

$$\Rightarrow \boxed{F_D(\hat{S}-1) \leq \frac{\beta}{\alpha+\beta} = \frac{\tilde{p}-c}{\tilde{p}+h} \leq F_D(\hat{S})} \quad \text{if } \hat{S}>0.$$

$$F_D(0) \geq \frac{\beta}{\alpha+\beta} = \frac{\tilde{p}-c}{\tilde{p}+h} \quad \text{if } \hat{S}=0.$$