Note that: $F_D(S) = P(D \le S) =$ Probability of no shortage in the period.

$F_D(S)$ is sometimes called the service level

Let $C_{under} = \tilde{p} - c$ = unit cost of underordering
= decrease in profit that results from failing to order a unit that could have been sold during the period

$C_{over} = c + h$ = unit cost of overordering
= decrease in profit that results from ordering a unit that could not be sold during the period.

Note $C_{under} + C_{over} = \tilde{p} + h$

$\Rightarrow F_D(S^*)$ = optimal service level $= \dfrac{C_{under}}{C_{under} + C_{over}}$

If $C_{under} = C_{over}$ then $F_D(S^*) = \frac{1}{2}$, i.e. probability of shortage is 50%.

Now we return to the case where D is a discrete random variable

$$C(S) = cS + h\sum_{d=0}^{S-1}(S-d)P_D(d) + \tilde{p}\sum_{d=S}^{\infty}(d-S)P_D(d)$$

Note: $c\sum_{d=0}^{S-1}(S-d)P_D(d) - c\sum_{d=S}^{\infty}(d-S)P_D(d) = c\sum_{d=0}^{\infty} S\, P_D(d) - c\sum_{d=0}^{\infty} d\,P_D(d)$

$= cS - cE[D]$

$$\Rightarrow C(S) = \underbrace{(c+h)}_{\alpha}\underbrace{\sum_{d=0}^{S-1}(S-d)P_D(d)}_{g(S)} + \underbrace{(\tilde{p}-c)}_{\beta}\underbrace{\sum_{d=S}^{\infty}(d-S)P_D(d)}_{f(S)} + cE[D].$$

constant independent of S

$= \alpha g(S) + \beta f(S)$ + constant.

Note: $\Delta g(S) = g(S+1) - g(S) = \sum_{d=0}^{S}(S+1-d)P_D(d) - \sum_{d=0}^{S-1}(S-d)P_D(d)$

$= \sum_{d=0}^{S} P_D(d) = P(D \le S) = F_D(S) \ge 0 \Rightarrow$ g increasing.

$\Delta^2 g(S) = \Delta g(S+1) - \Delta g(S) = \sum_{d=0}^{S+1} P_D(d) - \sum_{d=0}^{S} P_D(d) = P_D(S+1) \ge 0$

$\Rightarrow$ g integer-convex.

$\Delta f(S) = f(S+1) - f(S) = \sum_{d=S+1}^{\infty}(d-S-1)P_D(d) - \sum_{d=S}^{\infty}(d-S)P_D(d)$

$= -\sum_{d=S+1}^{\infty} P_D(d) = -P(D \ge S+1) = -(1-F_D(S)) \le 0 \Rightarrow$ f decreasing

$\Delta^2 f(S) = \Delta f(S+1) - \Delta f(S) = -\sum_{d=S+2}^{\infty} P_D(d) + \sum_{d=S+1}^{\infty} P_D(d) = P_D(S+1) \ge 0$

$\Rightarrow$ f integer-convex

Prop 3.1: $\hat{x}$ optimal iff $\frac{-\Delta f(\hat{x})}{\Delta g(\hat{x})} \le \frac{\alpha}{\beta} \le \frac{-\Delta f(\hat{x}-1)}{\Delta g(\hat{x}-1)}$ if $\hat{x} > 0$.

Here: $\hat{S}$ optimal iff $\frac{1-F_D(\hat{S})}{F_D(\hat{S})} \le \frac{\alpha}{\beta} = \frac{c+h}{\tilde{p}-c} \le \frac{1-F_D(\hat{S}-1)}{F_D(\hat{S}-1)}$

Reformulate:
$$\frac{1-F}{F} \leq \frac{\alpha}{\beta} \overset{F>0}{\iff} 1-F \leq \frac{\alpha}{\beta} F$$
$$\Updownarrow$$
$$\frac{1}{1+\frac{\alpha}{\beta}} = \boxed{\frac{\beta}{\alpha+\beta} \leq F} \iff 1 \leq \left(1+\frac{\alpha}{\beta}\right) F$$

$$\frac{\beta}{\alpha+\beta} = \frac{\tilde{p}-c}{c+h+\hat{p}-c} = \frac{\hat{p}-c}{\hat{p}+h}$$

compare $\begin{cases} \beta = C_{under} \\ \alpha = C_{over} \end{cases}$

$$\Rightarrow \boxed{F_D(\hat{S}-1) \leq \frac{\tilde{p}-c}{\hat{p}+h} \leq F_D(\hat{S})}$$