Deterministic Dynamic Programming

Consider a multistage problem

at each stage k the state s_k describes the current status

at each stage k a policy decision x_k is taken ($x_k \in \mathcal{F}_k(s_k)$ feasibility)

which determines the next state
$$s_{k+1} = h_k(s_k, x_k)$$

0	$\dots$	n	$n+1$	$\dots$	N
s_0	$\dots$	s_n	s_{n+1}	$\dots$	s_N
x_0	$\dots$	x_n	x_{n+1}	$\dots$	
h_0	$\dots$	h_n	h_{n+1}	$\dots$	

Consider the optimization problem :

$$(\mathcal{P}) \quad \left[\begin{array}{lll} \text{Minimize} & F(s_0, \dots, s_N, x_0, \dots, x_{N-1}) & \\ & & \\ \text{s.t.} & \left\{ \begin{array}{ll} s_{k+1} = h_k(s_k, x_k) & k = 0, 1, \dots, N-1 \\ x_k \in \mathcal{F}_k(s_k) & k = 0, 1, \dots, N-1 \\ s_0 = z_0 & \end{array} \right. & \end{array} \right]$$

We assume that F can be decomposed as

$$F(s_0, \dots, s_N, x_0, \dots, x_{N-1}) = \sum_{k=0}^{N-1} G_k(s_k, x_k)$$

or

$$F(s_0, \dots, s_N, x_0, \dots, x_{N-1}) = \prod_{k=0}^{N-1} G_k(s_k, x_k) \qquad \text{where } G_k \geq 0.$$

Let

$$(P_n) \quad \left[\begin{array}{lll} \text{Minimize} & F_n(s_n, \ldots, s_N, x_n, \ldots x_{N-1}) & \\ & & \\ \text{s.t.} & \left\{ \begin{array}{ll} s_{k+1} = h_k(s_k, x_k) & k = n, \ldots, N-1 \\ x_k \in \mathcal{F}_k(s_k) & k = n, \ldots, N-1 \\ s_n = s & \end{array} \right. & \end{array} \right]$$

Let $f_n^*(s)$ denote the optimal value of (P_n) with $s_n = s$.

$$F_n = \sum_{k=n}^{N-1} G_k$$

or

$$F_n = \prod_{k=n}^{N-1} G_k$$

Let

$$(P_n^z) \quad \left[\begin{array}{lll} \text{Minimize} & F_n(s_n, \ldots, s_N, x_n, \ldots, x_{N-1}) & \\ & & \\ \text{s.t.} & \left\{ \begin{array}{ll} s_{k+1} = h_k(s_k, x_k) & k = n, \ldots, N-1 \\ x_k \in \mathcal{F}_k(s_k) & k = n, \ldots, N-1 \\ s_n = s, \quad \underline{\underline{x_n = z}} & \end{array} \right. & \end{array} \right]$$

Let $f_n^*(s, z)$ denote the optimal value of (P_n^z) with $s_n = s$ and $x_n = z$

The optimal values are related by $f_n^*(s) = \min\limits_{z \in \mathcal{F}(s)} \{ f_n^*(s, z) \} = f_n^*(s, x_n^*)$

a recursive equation is given by

$$f_n^*(s, z) = G_n(s, z) \overbrace{*}^{+ \text{ or } \times} f_{n+1}^*(\overbrace{h_n(s, z)}^{= s_{n+1}})$$

Thus

$$f_n^*(s) = \min_{z \in \mathcal{F}_n(s)} \left\{ G_n(s, z) * f_{n+1}^*(h_n(s, z)) \right\}$$

often $f_N^*(s)$ is easy to determine, recursive relationship gives $f_{N-1}^*(s), \ldots, f_0^*(s)$, then $f_0^*(s)$ gives optimal value of (P)