Formula-sheet at the exam in SF2863, December 2009

No calculator at the exam!

If events happen according to a Poisson process with rate λ, τ denotes the time between two consecutive events, and $X(T)$ denotes the number of events on the time interval $[0,T]$, then
$P(\tau \le t) = 1 - e^{-\lambda t}$, $P(X(T) = \ell) = \dfrac{(\lambda T)^\ell}{\ell\,!} e^{-\lambda T}$, $\mathrm{E}[\tau] = 1/\lambda$, $\mathrm{E}[X(T)] = \lambda T$.

Markov chain in discrete time.
$\mathbf{P}$ = the matrix with elements $p_{ij} = P(X_{n+1} = j \mid X_n = i)$.
$\mathbf{p}^{(n)}$ = the row vector with components $p_j^{(n)} = P(X_n = j)$. Then $\mathbf{p}^{(n+1)} = \mathbf{p}^{(n)}\mathbf{P}$.
The row vector π defines a stationary distribution if $\pi = \pi \mathbf{P}$, $\sum_j \pi_j = 1$ and $\pi_j \ge 0$.

Markov chain in continuous time (also called Markov process with discrete state space).
$\mathbf{P}(h)$ the matrix with elements $p_{ij}(h) = P(X(t+h) = j \mid X(t) = i)$.
$\mathbf{p}(t)$ = the row vector with components $p_j(t) = P(X(t) = j)$. Then $\mathbf{p}(t+h) = \mathbf{p}(t)\mathbf{P}(h)$.
Assumption: $p_{ij}(h) = q_{ij}h + o(h)$ if $j \ne i$, while
$p_{ii}(h) = 1 + q_{ii}h + o(h) = 1 - q_i h + o(h)$, where $q_i = -q_{ii} = \sum_{j \ne i} q_{ij}$.
Thus, $\mathbf{P}(h) \approx \mathbf{I} + h\,\mathbf{Q}$ and $(\mathbf{p}(t+h) - \mathbf{p}(t))/h \approx \mathbf{p}(t)\mathbf{Q}$ for small $h > 0$.
The row vector π defines a stationary distribution if $\pi\mathbf{Q} = \mathbf{0}$, $\sum_j \pi_j = 1$ and $\pi_j \ge 0$.
The system $\pi\mathbf{Q} = \mathbf{0}$ can be written $\sum_{i \ne j} \pi_i q_{ij} + \pi_j q_{jj} = 0$, for all j, or
$\pi_j \sum_{k \ne j} q_{jk} = \sum_{i \ne j} \pi_i q_{ij}$ ("jumps out from state j = jumps into state j").

Some quantities and relations in queueing theory (where P_n corresponds to π_n above):
$L = \sum_{n=0}^{\infty} nP_n$, $L_q = \sum_{n=s}^{\infty} (n-s)P_n$, $\bar\lambda = \sum_{n=0}^{\infty} \lambda_n P_n$, $L = \bar\lambda W$, $L_q = \bar\lambda W_q$.
$M/M/1$: $\rho = \lambda/\mu < 1$, $P_0 = 1-\rho$, $P_n = \rho^n P_0$, $L = \dfrac{\rho}{1-\rho}$.
$M/M/2$: $\lambda_n = \lambda$ for $n \ge 0$, $\mu_1 = \mu$, $\mu_n = 2\mu$ for $n \ge 2$, $\rho = \lambda/(2\mu) < 1$,
$P_0 = \dfrac{1-\rho}{1+\rho}$, $P_n = 2\rho^n P_0$ for $n \ge 1$, $L = \dfrac{2\rho}{1-\rho^2}$.

Jackson queueing networks.
Calculate $\lambda_1, \ldots, \lambda_m$ from $\lambda_j = a_j + \sum_i \lambda_i p_{ij}$. Check $\lambda_j < s_j \mu_j$.
Analyze each service facility to obtain $P(N_j = n_j)$.
Then $P(N_1 = n_1, \ldots, N_m = n_m) = \prod_j P(N_j = n_j)$.
$W_1, \ldots, W_m$ can be obtained from $W_i = V_i + \sum_j p_{ij} W_j$, where $V_i = L_i/\lambda_i$.

Some deterministic inventory models.
EOQ with shortage not permitted: Minimize $\dfrac{Kd}{Q} + c\,d + \dfrac{hQ}{2}$.
$C_i = \min_j \{C_i^{(j)} \mid i \le j \le n\}$. $C_i^{(j)} = C_{j+1} + K + h(r_{i+1} + 2r_{i+2} + \cdots + (j-i)r_j)$.

Some stochastic inventory models.
$C(S) = c\,S + p\,\mathrm{E}[(\xi - S)^+] + h\,\mathrm{E}[(S-\xi)^+]$.
If ξ is a continuous non-negative random variable then
$\mathrm{E}[(\xi-S)^+] = \int_S^\infty (t-S) f_\xi(t)dt$, $\mathrm{E}[(S-\xi)^+] = \int_0^S (S-t) f_\xi(t)dt$,
and $C'(S) = c + p\,(F_\xi(S) - 1) + hF_\xi(S)$.
If ξ is a non-negative integer-valued random variable then S is integer and
$\mathrm{E}[(\xi-S)^+] = \sum_{j=S}^{\infty} (j-S) p_\xi(j)$, $\mathrm{E}[(S-\xi)^+] = \sum_{j=0}^{S} (S-j) p_\xi(j)$,
and $C(S+1) - C(S) = c + p\,(F_\xi(S) - 1) + hF_\xi(S)$.

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Marginal allocation for generating efficient solutions to the pair (f, g), where f and g are integer-convex separable functions, f decreasing and g increasing in the non-negative integer variables $x_1, \ldots, x_n$.
Generate a table in which the j:th column contains the quotients $-\Delta f_j(0)/\Delta g_j(0)$, $-\Delta f_j(1)/\Delta g_j(1)$, $-\Delta f_j(2)/\Delta g_j(2)$, ...
Let all the quotients in the table be uncanceled.
Initiate the variables to their smallest feasible values and repeat the following:
Let ℓ be the number of the column with the largest uncanceled quotient.
Cancel this quotient, and increase the ℓ:th variable x_ℓ by one.

Finite horizon MDP recursion (discounting if $0 < \alpha < 1$, no discounting if $\alpha = 1$):

$$V_i^{(n)} = \min_k \{ C_{ik} + \alpha \sum_j p_{ij}(k) V_j^{(n-1)} \} \quad \text{(backward time)}.$$

LP formulation for MDP without discounting:

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_i \sum_k C_{ik} y_{ik} \\ \text{subject to} \quad & \textstyle\sum_i \sum_k y_{ik} = 1, \\ & \textstyle\sum_k y_{jk} - \sum_i \sum_k p_{ij}(k) y_{ik} = 0, \ \text{for all } j, \\ & y_{ik} \geq 0, \ \text{for all } i \text{ and } k. \end{aligned}$$

Policy improvement algorithm for MDP without discounting:

1. For a given policy, calculate $v_0, \ldots, v_M$ and g from $v_M = 0$ and $g + v_i = C_{i,d_i} + \sum_j p_{ij}(d_i) v_j$.

2. The current policy is optimal if $g + v_i = \min_k \{ C_{ik} + \sum_j p_{ij}(k) v_j \}$.
Otherwise, define a new policy by letting $d_i =$ a minimizing k above.
Then go to 1.

LP formulation for MDP with discounting:

$$\begin{aligned} \text{minimize} \quad & \textstyle\sum_i \sum_k C_{ik} y_{ik} \\ \text{subject to} \quad & \textstyle\sum_k y_{jk} - \alpha \sum_i \sum_k p_{ij}(k) y_{ik} = \beta_j, \ \text{for all } j, \\ & y_{ik} \geq 0, \ \text{for all } i \text{ and } k. \end{aligned}$$

where the constants in the right hand sides should satisfy $\beta_j > 0$ and $\sum_j \beta_j = 1$.

Policy improvement algorithm for MDP with discounting:

1. For a given policy, calculate $V_0, \ldots, V_M$ from $V_i = C_{i,d_i} + \alpha \sum_j p_{ij}(d_i) V_j$.

2. The current policy is optimal if $V_i = \min_k \{ C_{ik} + \alpha \sum_j p_{ij}(k) V_j \}$.
Otherwise, define a new policy by letting $d_i =$ a minimizing k above.
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