Home assignment number 1, 2012, in SF2863 Systems Engineering.

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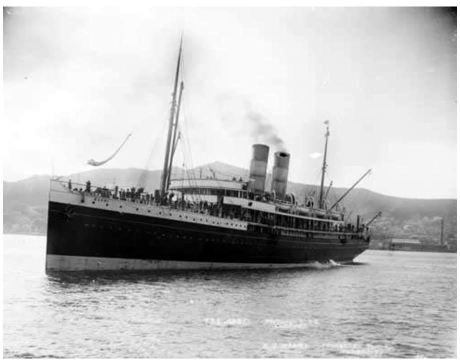
In this home assignment it is allowed to cooperate in groups of at most three persons each. Limited cooperation with other groups is also allowed, but should be refered to in the report.
One report per group is to be handed in, and you should describe in your own words how the problem was solved. Make the description rich enough so that it is totally clear how you defined your states and modelled the problem and describe how you implemented the simulations. It is not allowed to copy parts of another groups report or computer code! The answers to the questions in the assignment should be given in the main report. (Not in your computer printouts.) Relevant print-outs and plots should be included in the report. Computer code should be held available on request.
State your name, personal number and email adress on the front of the report.

A written report (printed out on paper) should be handed in to the instructor before 15.15, Tuesday, November 6. Note that this deadline is sharp. Directly after there will be oral presentations. Some students will be randomly selected for oral presentation of the assignment at the beginning of the exercise session, so it is necessary to attend that session to get all the bonus points. (Note that the assignment is completely voluntary, but that a correct report that is handed in in time will grant you bonus points at the final exam.)

This home assignment can grant a maximum of two bonus points on the exam.

In this home assignment a simple Markov process in continuous time will be examined.
We consider a big (old) ferry with two engines that each drives one propeller. If one engine breaks down the ferry can still move, but it runs a bit slower, and if both breaks down the ferry stops.



Assume that engine number 1 breaks down after some time that is modelled as a stochastic variable with $\text{Exp}(\lambda_1)$ distribution. Similarly, engine number 2 breaks down after some time that is $\text{Exp}(\lambda_2)$ distributed. Three repairmen are working on the ferry for repairing the engines. The repair time is $\text{Exp}(n_k\mu)$ distributed, where n_k is the number of repairmen working on engine k.
Let $V(t)$ denote the speed of the ferry at time t. If both the engines are working the ferry is running at a speed which is v, if only engine 1 is working the speed is v_1 and if only engine 2 is working the speed is v_2.

Now, the system is a Markov process that will depend on how many workers are assigned to repair the engines if they break down. If only one engine is broken it is obvious that it is optimal to let all three repairmen work on it. If both engines are broken it is not so clear how to distribute the workers. To decide this you should for each different repair strategy determine the average speed of the ferry at steady state and this way choose the best strategy.

You should do this in two different ways.

First, all of you should determine the average speed using the theory of Markov processes. For this approach you need to define the states of the Markov process, the intensity matrix, motivate why there should exist a stationary distribution and determine it. The average speed can then be determined from this stationary distribution.

Second you should simulate the system numerically and determine the average speed by ergodic estimates.

This will be done in two different ways; half of the groups will use a time discretization approach, while the second half will directly simulate the continuous time process.

If the oldest group member is born on an even day, then the whole group should use the time discretization approach, and if the oldest member is born on an odd day the group should simulate in continuous time.

Discretization approach

Now a discretization of the continuous time process will be made.

Start the process at time 0 with both engines of the ferry functioning. Then make a discretization of the time axis so that we only consider the process at times $t_k = kh$ for $k = 0, 1, 2, \cdots, N$. Then the probability of a jump to another state during the time interval $[t_k,\ t_{k+1}]$ can be determined (approximatively) using the intensity matrix for the continuous time Markov process. (Without explicitly using the exponential distributions.) If the time step h is small this approximation is good. These probabilities form the transition matrix of a discrete time Markov chain that will approximate the continuous time one. Check if the transition matrix you obtain has row sums equal to one.

Now use this discretization to simulate the process. If you use Matlab, a stochastic variable with uniform probability in $[0,\ 1]$ can be generated using the command `rand`. Then take averages to determine estimates of the expected speed of the ferry.

The approach just described has also the purpose of showing the connection between discrete time and continuous time Markov chains.

Continuous time approach

It is of course also possible to simulate the continuous time process. Start the process at time 0 with both engines of the ferry functioning. Then you should use that the jump times are exponentially distributed. If you use Matlab an exponentially distributed variable can be generated using the command `exprnd`, where you specify the mean of the variable (which is inversely proportional to the intensity). Then you use the exponential distribution to determine how long you remain in a state. You can use several exponential distributions, or one exponential distribution and a uniform probability generated using the command `rand`, to determine how long you remain in a state and which will be the next state. Run the simulation for a time interval $[0,\ T]$.

Theory question

A last exercise is given in the box below:

If τ is the stochastic variable that describes the time to the next jump of the continuous time Markov process $X(t)$ with intensity q, then we know that τ must be $\text{Exp}(q)$ distributed. That means that τ has the distribution function $F_\tau(t) = P(\tau \leq t) = 1 - e^{-qt}$.
Show that
if τ has the $\text{Exp}(q)$ distribution, then $P(\tau \geq 1) = P(\tau \geq 1.5 \mid \tau \geq 0.5)$.
This is a property any time-homogeneous Markov process should have, why ?

Parameter values

- $v = 18$, $v_1 = 10$, and $v_2 = 12$.
- $\lambda_1 = 10$, $\lambda_2 = 15$, and $\mu = 8$
- $h = 0.001$
- N and T are large enough. Try to find good values yourselves. They have to be rather large to give good ergodic estimates.

Hints
Since a large number of time steps have to be made to get reasonable ergodic estimates you might run into memory problems if you store all the old states in the simulations. Please note that it is enough to keep track on how long time the Markov chain has spent so far in each state to determine the estimates for the stationary probabilities and the average speed.

Good luck!