1. Slightly revised notations for part of Lecture 2, by Krister Svanberg

Consider a system with $M+1$ states called $0, 1, ..., M$.

Let $X(t) =$ the state of the system at time t. $X(t)$ is a random variable, defined for $t \geq 0$.

Assumption 1: If $0 \leq s < t \leq t+h$, then

$$P(X(t{+}h) = j \mid X(t) = i,\ X(s) = \ell\,) \;=\; P(X(t{+}h) = j \mid X(t) = i). \tag{1.1}$$

In words, "$P(X(t{+}h) = j \mid X(t) = i)$ does not depend on the history before time t".

Assumption 2: If $0 \leq t \leq t+h$, then

$$P(X(t{+}h) = j \mid X(t) = i) \;=\; P(X(h) = j \mid X(0) = i). \tag{1.2}$$

In words, "$P(X(t{+}h) = j \mid X(t) = i)$ does not depend on t".

Let

$$p_{ij}(h) = P(X(h) = j \mid X(0) = i), \tag{1.3}$$

and let $\mathbf{P}(h)$ be the $(M{+}1)\times(M{+}1)$ matrix with elements $p_{ij}(h)$. Note that $\mathbf{P}(0) = \mathbf{I}$.

Assumption 3: There are constants $q_{ij} \geq 0$, $i \neq j$, and $q_{ii} \leq 0$, such that

$$p_{ij}(h) = q_{ij}h + o(h) \;\text{ if } j \neq i, \;\text{ while }\; p_{ii}(h) = 1 + q_{ii}h + o(h). \tag{1.4}$$

Since $\sum_j p_{ij}(h) = 1$, we get that $\sum_j q_{ij} = 0$, so that $q_{ii} = -\sum_{j\neq i} q_{ij}$ for all i.

Sometimes the notation $q_i = -q_{ii}$ is used, which means that

$$p_{ii}(h) = 1 - q_i h + o(h)\,, \;\text{ where }\; q_i = \sum_{j\neq i} q_{ij} \geq 0. \tag{1.5}$$

Assumption 3 is equivalent to that

$$\lim_{h\to 0^+} \frac{\mathbf{P}(h) - \mathbf{I}}{h} = \mathbf{Q}\,, \tag{1.6}$$

where $\mathbf{Q}$ is the $(M{+}1)\times(M{+}1)$ matrix with elements q_{ij}.
Now let

$$p_j(t) = P(X(t) = j) \;\text{ and }\; \mathbf{p}(t) = (p_0(t), ...p_M(t)). \tag{1.7}$$

Then, for all $t \geq 0$ and $h \geq 0$,

$$p_j(t{+}h) = \sum_{i=0}^{M} p_i(t)\, p_{ij}(h), \tag{1.8}$$

which equivalently can be written

$$\mathbf{p}(t{+}h) = \mathbf{p}(t)\mathbf{P}(h). \tag{1.9}$$

By subtracting $\mathbf{p}(t)$ from both sides and dividing by h, we get

$$\frac{\mathbf{p}(t{+}h) - \mathbf{p}(t)}{h} = \mathbf{p}(t)\,\frac{\mathbf{P}(h) - \mathbf{I}}{h}, \tag{1.10}$$

and by letting $h \to 0^+$, we obtain

$$\dot{\mathbf{p}}(t) = \mathbf{p}(t)\mathbf{Q}. \tag{1.11}$$

If $\mathbf{p}(0)$ is known, this system of linear differential equations can be solved to obtain $\mathbf{p}(t)$.

Example:

Consider a simple system with only two states:
"Functioning" (= state 0) and "Failed" (= state 1).
Let $X(t)$ = the state of the system at time t.

Assumptions:

$P(X(t+h) = 1 \mid X(t) = 0) = \lambda h + o(h)$,
$P(X(t+h) = 0 \mid X(t) = 0) = 1 - \lambda h + o(h)$,
$P(X(t+h) = 0 \mid X(t) = 1) = \mu h + o(h)$,
$P(X(t+h) = 1 \mid X(t) = 1) = 1 - \mu h + o(h)$.

Interpretation:
When the system is functioning, the time until it fails is $\exp(\lambda)$.
When the system is failed, the time until it will function is $\exp(\mu)$.
The expected functioning time between failures is $1/\lambda$,
The expected repair time is $1/\mu$.
λ is the *failure rate* while μ is the *repair rate*.

The matrix $\mathbf{Q}$ for this example is $\mathbf{Q} = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$.

Let $p_0(t) = P(X(t) = 0)$ and $p_1(t) = P(X(t) = 1)$.

Then the differential equations $\dot{\mathbf{p}}(t) = \mathbf{p}(t)\mathbf{Q}$ becomes

$$\begin{aligned} \dot{p}_0(t) &= -\lambda\, p_0(t) + \mu\, p_1(t), \\ \dot{p}_1(t) &= \ \ \lambda\, p_0(t) - \mu\, p_1(t). \end{aligned}$$

The solution of this system is

$$\begin{aligned} p_0(t) &= \frac{\mu}{\lambda+\mu} + \left(p_0(0) - \frac{\mu}{\lambda+\mu}\right) e^{-(\lambda+\mu)\,t}, \\ p_1(t) &= \frac{\lambda}{\lambda+\mu} + \left(p_1(0) - \frac{\lambda}{\lambda+\mu}\right) e^{-(\lambda+\mu)\,t}, \end{aligned}$$

from which it follows that

$$p_0(t) \to \frac{\mu}{\lambda+\mu} = \pi_0 \ \text{ and } \ p_1(t) \to \frac{\lambda}{\lambda+\mu} = \pi_1, \quad \text{when } t \to \infty.$$

Note that this aymptotic distribution $\pi = (\pi_0, \pi_1)$ is the unique solution to the system of linear equations

$$\pi\,\mathbf{Q} = \mathbf{0} \ \text{ and } \ \pi_0 + \pi_1 = 1.$$

This is no coincidence!