KTH Mathematics

Suggested solutions for the exam in SF2863 Systems Engineering.
December 18, 2010 8.00–13.00

Examiner: Per Enqvist, phone: 790 62 98

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1. (a) Introduce the two states, $s = e$, if the parking spot is empty, and $s = f$, if it is full.

Let the decision $k = 1$ be that you take the parking spot (if possible) and $k = 2$ be that you do not take it.

Let the parking spots be numbered from $-T$ to T. Let stage n be that you are at parking spot n.

Define the optimal value function $V_n^*(s)$ be the expected value if you are standing at parking spot n, the state of that parking spot is s and you use optimal parking decisions.

Define also the function $V_n(s, k)$ to be the expected value if you are standing at parking spot n, the state of that parking spot is s you make first decision k and then you use optimal parking decisions.

If $s = f$, you can not make the decision to park there, so clearly $V_n^*(f) = V_n(s, 2) = 0.5V_{n+1}^*(e) + 0.5V_{n+1}^*(f)$. If $s = e$, you have the option to park there, so clearly $V_n^*(e) = \min_k\{V_n(s, k)\} = \min\{|n|, 0.5V_{n+1}^*(e) + 0.5 * V_{n+1}^*(f)\}$.

To initiate the recursion, note that if you are standing at the last parking spot and it is full, you have to leave without getting a parking spot and the value is then M, *i.e.* $V_T^*(f) = M$. If the last parking spot is empty, the value is M if you do not take it and T if you do take it, since $M > T$ you take it and $V_T^*(e) = T$.

(b) Use the recursion.

First $V_2(e) = 2$ and $V_2(f) = 5$.

Then $V_1^*(f) = 0.5V_2^*(e) + 0.5V_2^*(f) = 0.5 \cdot 2 + 0.5 \cdot 5 = 3.5$,
and $V_1^*(e) = \min\{1,\ .5V_2^*(e) + 0.5V_2^*(f)\} = \min\{1,\ 3.5\} = 1$ for the decision $k = 1$, *i.e.*, to park.

Then $V_0^*(f) = 0.5V_1^*(e) + 0.5V_1^*(f) = 0.5 \cdot 1 + 0.5 \cdot 3.5 = 2.25$,
and $V_0^*(e) = \min\{0,\ .5V_1^*(e) + 0.5V_1^*(f)\} = \min\{0,\ 2.25\} = 0$ for the decision $k = 1$, *i.e.*, to park.

Then $V_{-1}^*(f) = 0.5V_0^*(e) + 0.5V_0^*(f) = 0.5 \cdot 0 + 0.5 \cdot 2.25 = 1.125$,
and $V_{-1}^*(e) = \min\{|-1|,\ .5V_0^*(e) + 0.5V_0^*(f)\} = \min\{1,\ 1.125\} = 1$ for the decision $k = 1$, *i.e.*, to park.

Then $V_{-2}^*(f) = 0.5V_{-1}^*(e) + 0.5V_{-1}^*(f) = 0.5 \cdot 1 + 0.5 \cdot 1.125 = 1.0625$,
and $V_{-2}^*(e) = \min\{|-2|,\ 0.5V_{-1}^*(e) + 0.5V_{-1}^*(f)\} = \min\{2,\ 1.0625\} = 1$ for the decision $k = 2$, *i.e.*, not to park.

The optimal strategy is to not park in the first spot, and then to park in the first empty spot coming up.

It should be noted that this approach can be used to solve problems where e.g. you receive job offers in sequence and you have to decide if to take the job or continue searching for a better one.

2. (a) This is the basic EOQ model with $d = 10$ kilo per hour, $c = 5$ Euro per kilo, $h = 0.00002$ Euro per kilo and hour ans $K = 100$ Euro.

Then the optimal order quantity is given by $Q^* = \sqrt{\frac{2dK}{h}} = \sqrt{\frac{2\cdot 10\cdot 100}{2\cdot 10^{-5}}} = 10^4$ kilos.

Frasse should order this with a time period of $t = Q^*/d = 1000$ hours, *i.e.* nearly 42 days.

(b) If Frasse does not allow shortage, he should buy 1900 kilo of coffee per week. The expected cost is $C(1900) =$ determined below.

(c) If Frasse allows shortage, then this is a standard stochastic single period model with no ingoing inventory.

The cost $c = 5$, shortage cost is $p = 20$ and $h = -2$ is the holding cost (salvage value).

Then $C(D, S) = cS + p(D - S)^+ + h(S - D)^+$ is the cost for a specific demand D and then

$$C(S) = cS + p\int_S^\infty (t - S)f_D(t)dt + h\int_0^S (S - t)f_D(t)dt$$

is the expected value of this cost where we have used the probability density function $f_D(d)$ of D. Now if $C'(S) = c + p(F_D(S) - 1) + hF_D(S) = 0$ at some point it must be the optimum, since the function is convex. That is, we should solve the equation

$$F_D(S^*) = \frac{p - c}{p + h} = \frac{15}{18}.$$

Here,

$$F_D(S^*) = \int_0^{S^*} f_D(t)dt = \int_{1500}^{S^*} \frac{1}{400} = \frac{S^* - 1500}{400}$$

if $S^* \in [1500, 1900]$. Hence, $S^* = 1500 + 400\frac{15}{18}$.

The expected cost is now

$$C(S^*) = cS^* + \frac{20}{400}\int_{S^*}^{1900} (t - S^*)dt + \frac{-2}{400}\int_{1500}^{S^*} (S^* - t)dt.$$

$$C(S^*) = cS^* + \frac{20}{400}\left[t^2/2 - tS^*\right]_{t=S^*}^{t=1900} + \frac{-2}{400}\left[tS^* - t^2/2\right]_{1500}^{S^*}$$

$$C(S^*) = cS^* + \frac{20}{400}\left(1900^2/2 - 1900S^* - (S^*)^2/2 + (S^*)^2\right)$$

$$+\frac{-2}{400}\left((S^*)^2 - (S^*)^2/2 - 1500S^* + 1500^2/2)\right)$$

$$C(S^*) = cS^* + \frac{20}{400}\left(1900 - S^*\right)^2/2 + \frac{-2}{400}\left(S^* - 1500\right)^2/2 = 9000$$

In particular, the optimal cost if shortage is not allowed, the answer to (b), is given by

$$C(1900) = 1900 \cdot 5 - \frac{2}{400}\left(1900 - 1500\right)^2/2 = 9100.$$

$C(S^*)$ is the minimum of C so it is smaller than $C(1900)$, so Frasse should use the strategy to pay off the customers.

3. Define the states $s = 1$ to signify that the atmosphere is great, and the state $s = 2$ to signify that the atmosphere is good.

Define the decision $k = 1$ to signify that fresh coffee will be served and $k = 2$ to signify that budget coffee is served.

The transition probabilities are then given by

$$p_{11}(1) = 0.9, \quad p_{12}(1) = 0.1, \quad p_{21}(1) = 0.9, \quad p_{22}(1) = 0.1,$$

$$p_{11}(2) = 0.5, \quad p_{12}(2) = 0.5, \quad p_{21}(2) = 0.5, \quad p_{22}(2) = 0.5.$$

The immediate costs C_{ik}, being in state i making decision k, are given by
$C_{11} = -40 + 10 = -30$ *increased revenue + cost of fresh coffee*
$C_{12} = -40$ *increased revenue*
$C_{21} = 10$ *cost of fresh coffee*
$C_{22} = 0$
Let $y = [y_{11}\ y_{12}\ y_{21}\ y_{22}]^T$.
The objective function of the LP is determined by $c = [C_{11}\ C_{12}\ C_{21}\ C_{22}]^T$.
The constraints corresponds to the equations

$$y_{11} + y_{12} - \frac{1}{2}\left(p_{11}(1)y_{11} + p_{11}(2)y_{12} + p_{21}(1)y_{21} + p_{21}(2)y_{22}\right) = \beta_1$$

$$y_{21} + y_{22} - \frac{1}{2}\left(p_{12}(1)y_{11} + p_{12}(2)y_{12} + p_{22}(1)y_{21} + p_{22}(2)y_{22}\right) = \beta_2$$

which corresponds to

$$A = \begin{bmatrix} 1 - 0.5 \cdot 0.9 & 1 - 0.5 \cdot 0.5 & -0.5 \cdot 0.9 & -0.5 \cdot 0.5 \\ -0.5 \cdot 0.1 & -0.5 \cdot 0.5 & 1 - 0.5 \cdot 0.1 & 1 - 0.5 \cdot 0.5 \end{bmatrix}, \quad b = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix},$$

if $\beta_1 = \beta_2 = 1/2$.
From $D_{ik} = y_{ik}/(y_{i1} + y_{i2})$ we see that $D_{ik} = y_{ik}$ in this case and therefore, in state 1 the optimal decision is 2 and in state 2 the optimal decision is 2.
The optimal value is $y^T c = -40$.

(b) Start with the optimal policy from (a), i.e. let $d_1(R_1) = 2$ and $d_2(R_1) = 2$.
First step of the policy improvement algorithm is to determine V_1 and V_2 from the value determination equation.

$$V_1 = C_{12} + 0.5\left(p_{11}(2)V_1 + p_{12}(2)V_2\right) = -40 + 0.5(0.5V_1 + 0.5V_2)$$

$$V_2 = C_{22} + 0.5\left(p_{12}(2)V_1 + p_{22}(2)V_2\right) = 0 + 0.5(0.5V_1 + 0.5V_2)$$

We get $V_1 = -60$ and $V_2 = -20$.
To find out if it is optimal we do the policy iteration.
For $i = 1$

$\min_k C_{1k}+\alpha(p_{11}(k)V_1+p_{12}(k)V_2) = \min\{C_{11}+\alpha(p_{11}(1)V_1+p_{12}(1)V_2), C_{12}+\alpha(p_{11}(2)V_1+p_{12}(2)V_2)\}$

$\min\{-30+0.5(0.9(-60)+0.1(-20)), -40+0.5(0.5(-60)+0.5(-20))\} = \min\{-58, -60\}$

so the minimizing $\hat{k}_1 = 2$.
For $i = 2$

$\min_k C_{2k}+\alpha(p_{21}(k)V_1+p_{22}(k)V_2) = \min\{C_{21}+\alpha(p_{21}(1)V_1+p_{22}(1)V_2), C_{22}+\alpha(p_{21}(2)V_1+p_{22}(2)V_2)\}$

$\min\{10+0.5(0.9(-60)+0.1(-20)), 0+0.5(0.5(-60)+0.5(-20))\} = \min\{-18, -20\}$

so the minimizing $\hat{k}_2 = 2$.
The optimal value corresponding to the LP is $\beta_1 V_1 + \beta_2 V_2 = 0.5(-60 = +0.5(-20) = -40$,

(c) We repeat the calculations, now with the policy improvement algorithm without discounting. Start with R_1 as above.
The value determination equation is then

$$g + v_1 = c_{12} + p_{11}(2)v_1 + p_{12}(2)v_2$$

$$g + v_2 = c_{22} + p_{21}(2)v_1 + p_{22}(2)v_2$$

Assuming that $v_2 = 0$,

$$g + v_1 = -40 + 0.5v_1$$

$$g = 0 + 0.5v_1$$

gives $g = -20$ and $v_1 = -40$.
To find out if it is optimal we do the policy iteration.
For $i = 1$

$\min_k C_{1k}+p_{11}(k)v_1+p_{12}(k)v_2) = \min\{C_{11}+p_{11}(1)v_1+p_{12}(1)v_2, C_{12}+p_{11}(2)v_1+p_{12}(2)v_2\}$

$\min\{-30 + 0.9(-40) + 0.1(0)), -40 + 0.5(-40) + 0.5(0))\} = \min\{-66, -60\}$

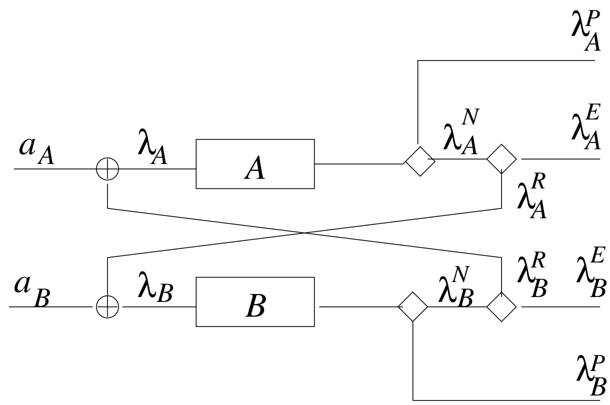
so the minimizing $\hat{k}_1 = 1$.
For $i = 2$

$\min_k C_{2k}+p_{21}(k)v_1+p_{22}(k)v_2 = \min\{C_{21}+p_{21}(1)v_1+p_{22}(1)v_2, C_{22}+p_{21}(2)v_1+p_{22}(2)v_2\}$

$\min\{10 + 0.9(-40) + 0.1(0)), 0 + 0.5(-40) + 0.5(0)\} = \min\{-26, -20\}$

so the minimizing $\hat{k}_2 = 1$.
So the policy R_1 is not optimal if there is no discounting.

4. We can think of the parking situation as a Jackson network,



where $a_A = 470$, $a_B = 940$.

The cruising around for finding a parking can be seen as a $M|M|\infty$ queueing system with service intensities given by the parking behavior, i.e. the parking intensity is $60/10 = 6$ (per hour) and the giving up intensity is $60/15 = 4$ (per hour).

The time until the minimum of the two exponential distributed events occurs is also exponentially distributed, but with intensity $6 + 4 = 10$ (per hour).

From the property of the exponential distribution we have a disaggregation of the process and intensity λ_A divides into λ_A^P (for those who find parking) and λ_A^N (for those who leave the lot without parking) where $\lambda_A^P = 6/(6+4)\lambda_A$ and $\lambda_A^N = 4/(6+4)\lambda_A$. Similarly for parking lot B, $\lambda_B^p = 6/(6+4)\lambda_B$ and $\lambda_B^N = 4/(6+4)\lambda_B$.

Furthermore, $\lambda_A^E = 1/2\lambda_A^N$ (intensity for those exiting A and going home), $\lambda_A^R = 1/2\lambda_A^N$ (intensity for those returning to lot B) and $\lambda_B^E = 1/4\lambda_B^N$, $\lambda_B^R = 3/4\lambda_B^N$.

(a) We model this situation as a Jackson network.
Balance equations

$$\lambda_A = a_A + \lambda_B^R = a_A + 3/4\lambda_B^N = a_A + 3/4 \cdot 4/10\lambda_B$$

$$\lambda_B = a_B + \lambda_A^R = a_B + 1/2\lambda_A^N = a_B + 1/2 \cdot 4/10\lambda_A$$

that is

$$470 = \lambda_A - 3/10\lambda_B$$

$$940 = \lambda_B - 2/10\lambda_A$$

which gives $\lambda_A = 800$ and $\lambda_B = 1100$.
The fraction between those who eventually finds a parking spot and those who go home without parking is

$$\frac{\lambda_A^P + \lambda_B^P}{\lambda_A^E + \lambda_B^E} = \frac{6/10\lambda_A + 6/10\lambda_B}{1/2\lambda_A^N + 1/4\lambda_B^N} = \frac{6/10 \cdot 800 + 6/10 \cdot 1100}{1/2 \cdot 4/10 \cdot 800 + 1/4 \cdot 4/10 \cdot 1100} = 38/9.$$

So almost 4 times as many finds parking compared to those who leave without.

(b) Let V_A and V_B be the average time it takes from a car enters one of the parking lots until it either finds a parking spot or gets tired and leaves it. We know that the intensity for this is 10, so the average time is $60/10 = 6$ minutes.

Letting W_A be the average time from a car arrives to parking lot A until it leaves the system (either parks or goes home), and W_B be the average time from a car arrives to parking lot B until it leaves the system, then

$$W_A = V_A + 2/10W_B = 6 + 2/10W_B$$

$$W_B = V_B + 3/10W_A = 6 + 3/10W_A$$

and $W_A = 360/47$ and $W_B = 390/47$.

An arbitrary car arrives at parking lot A with probability $a_A/(a_A + a_B) = 1/3$ and at parking lot B with probability $a_B/(a_A+a_B) = 2/3$, so the average time for an arbitrary car is

$$\frac{a_A}{a_A + a_B}W_A + \frac{a_B}{a_A + a_B}W_B = \frac{1}{3}\frac{360}{47} + \frac{2}{3}\frac{390}{47} = \frac{380}{47}$$

minutes.

(c) The number of cars cruising around is the expected number of “customers” in queueing system A. The average waiting time is $W = V_A = 6$ minutes and the “service” intensity is 10, so by Little’s formula $L = \lambda_A W = 800 \cdot 1/10 = 80$.

Alternatively, we can think of the parking lot as the repair shop at a base with expected service time equal 6 minutes, and approximate the arrivals as a Poisson process with intensity $\lambda_A = 800$ (per hour). Then, according to Palm’s Theorem the number of units in the repair shop has a Poission distribution with mean $\lambda_A T = 800 \cdot 6/60 = 80$.

5. (a) We were given

TABLE 1

n	$p_1(n)$	$p_2(n)$	$p_3(n)$
1	20	30	40
2	15	20	25
3	13	16	20
4	11	12	17
5	10	10	15

.

Then taking differences

TABLE 2

n	$\Delta p_1(n)$	$\Delta p_2(n)$	$\Delta p_3(n)$
1	-5	-10	-15
2	-2	-4	-5
3	-2	-4	-3
4	-1	-2	-2

and again

TABLE 3

n	$\Delta^2 p_1(n)$	$\Delta^2 p_2(n)$	$\Delta^2 p_3(n)$
1	3	6	10
2	0	0	2
3	1	2	1

and since $\Delta^2 p_i(n) \geq 0$ for all i and n the function p is integer-convex. It is seperable since it can be written as the sum of functions only depending on one element in n each.

(b) From (a) we know that p is a seperable integer-convex function, from the second table we see that it is also decreasing. Let $f = p$ and $g(n) = n_1 + n_2 + n_3$, where g is now increasing and integer-convex, since $\Delta^2 g = 0$.

We can now use the marginal allocation algorithm, making the table with columns defined by $-\Delta p_i(n)/\Delta g_i(n) = -\Delta p_i(n)$ since $\Delta g_i(n) = 1$:

TABLE MA

n	$-\Delta p_1(n)$	$-\Delta p_2(n)$	$-\Delta p_3(n)$
1	5	10	15
2	2	4	5
3	2	4	3
4	1	2	2

The largest element is 15, so $n^{(4)} = (1, 1, 2)$ is the optimal allocation for 4 researchers and $p(n^{(4)}) = 75$.

The largest element is 10, so $n^{(5)} = (1, 2, 2)$ is the optimal allocation for 5 researchers and $p(n^{(5)}) = 75 - 10 = 65$.

The largest element is 5, so $n^{(6)} = (2, 2, 2)$ or $n^{(6)} = (1, 2, 3)$ is the optimal allocation for 6 researchers and $p(n^{(8)}) = 65 - 5 = 60$.

The largest element is again 5, so $n^{(7)} = (2, 2, 3)$ is the optimal allocation for 7 researchers and $p(n^{(7)}) = 60 - 5 = 55$.

The largest element is 4, so $n^{(8)} = (2, 3, 3)$ is the optimal allocation for 8 researchers and $p(n^{(8)}) = 55 - 4 = 51$.