KTH Mathematics

Suggested solutions for the exam in SF2863 Systems Engineering.
June 9, 2011 14.00–19.00

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1. (a) Introduce the state, x_ℓ = how many dollars Frasse has day ℓ. We know that $x_0 = M$, and $x_\ell \in \{0, 1, 2, \cdots, \geq K\}$, where the last state indicates that Frasse has enough money to pay the loan-shark and will not play anymore, i.e. it is an absorbing state. The states 0,1,2 could also be collapsed to one state, assuming that $K \geq 3$, since if Frasse has 2 dollars or less he can never get more than 2 dollars by playing the game.

Let c_ℓ = how many dollars Frasse bets day ℓ. We know that $c_\ell \in \{0, \cdots, x_\ell - 1\}$. If $c_\ell = 0$ then $x_{\ell+1} = x_\ell$ and if $c_\ell = k > 0$ then $x_{\ell+1} = x_\ell - 1 + k$ with probability 0.6 and $x_{\ell+1} = x_\ell - 1 - k$ with probability 0.4.

Define the value function $V_\ell^*(x)$ to be the probability that Frasse can pay back the loan-shark on day n given that at day ℓ he has x dollars and use the optimal betting strategy.

The DynP equation becomes

$$V_\ell^*(x) = \max_{k \in 0, \cdots, x-1} \left\{ V_{\ell+1}^*(x), 0.6V_{\ell+1}^*(x - 1 + k) + 0.4V_{\ell+1}^*(x - 1 - k) \right\}$$

The boundary condition is that $V_n^*(x) = 1$ if $x \geq K$ and 0 otherwise.

(b) Use the recursion.

First $V_2(x) = 1$ if $x \geq 6$ and 0 otherwise.

Then $V_1^*(\geq 6) = \max\{V_2^*(\geq 6), 0.6V_2^*() + 0.4V_2^*()\} = 1$ for $c_1 = 0$,
$V_1^*(5) = \max\{V_2^*(5), 0.6V_2^*(\geq 6) + 0.4V_2^*(2), 0.6V_2^*(\geq 6) + 0.4V_2^*(1)\} = 0.6$ for $c_1 = 2, 3$,
$V_1^*(4) = \max\{V_2^*(4), 0.6V_2^*(5) + 0.4V_2^*(1), 0.6V_2^*(\geq 6) + 0.4V_2^*(0)\} = 0.6$ for $c_1 = 3$,
$V_1^*(3) = \max\{V_2^*(3), 0.6V_2^*(4) + 0.4V_2^*(0)\} = 0$ for all feasible c_1,
$V_0^*(2) = 0$.

Then $V_0^*(\geq 6) = \max\{V_2^*(\geq 6), 0.6V_2^*() + 0.4V_2^*()\} = 1$ for $c_1 = 0$,
$V_0^*(5) = \max\{V_1^*(5), 0.6V_1^*(\geq 6) + 0.4V_1^*(2), 0.6V_1^*(\geq 6) + 0.4V_1^*(1)\} = 0.6$ for $c_0 = 0, 2, 3$,
$V_0^*(4) = \max\{V_1^*(4), 0.6V_1^*(5) + 0.4V_1^*(1), 0.6V_1^*(\geq 6) + 0.4V_1^*(0)\} = 0.6$ for $c_0 = 0, 3$,
$V_0^*(3) = \max\{V_1^*(3), 0.6V_1^*(4) + 0.4V_1^*(0)\} = 0.36$ for $c_0 = 2$,
$V_0^*(2) = 0$.

If $M = 1, 2$ then there is no strategy that can save Frasse.

If $M = 3$, then Frasse should bet 2 dollars the first day and if he wins he has 4 dollars and should bet 3 dollars day 2.

If $M = 4$, then Frasse should bet 0 or 3 dollars the first day. If he bets and wins he has 6 dollars and is safe, if he does not bet the first day he bets 3 dollars day 2.

If $M = 5$, then Frasse should bet 0, 2 or 3 dollars the first day. If he bets and wins he has ≥ 6 dollars and is safe, if he does not bet the first day he bets 2 or 3 dollars day 2.

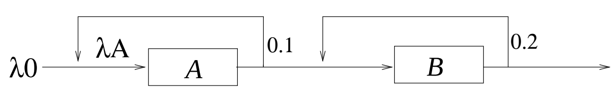
If $M = 6$ he should make no bets and is safe.

2. (a) This is the basic EOQ model with $d = 1$ kilo per week, $c = 4 - 6$ kSEK per kilo, $h = 0.1$ kSEK per kilo and week and $K = 20$ kSEK.

Then the optimal order quantity is given by $Q^* = \sqrt{\frac{2dK}{h}} = \sqrt{\frac{2 \cdot 1 \cdot 20}{0.1} = 20}$ kilos. Frasse should order this with a time period of $t = Q^*/d = 20/1 = 20$ weeks, and that is the time between his visits to Russia.

(b) EOQ model with quantity discounts. The total cost per unit time, if the unit cost is c_j, is $T_j(Q) = \frac{dK}{Q} + dc_j + \frac{hQ}{2}$. Both the curves T_1 and T_2 are convex and has a zero derivative for $Q^* = 20$. The cost $c_1 = 4 - 6 = -2$ is valid for $0 \leq Q < 50$, and The cost $c_2 = 2 - 6 = -4$ is valid for $Q \geq 50$, so we should compare the costs $T_1(20) = 0$ and $T_2(50) = -1.1$. The optimal order quantity will then change to 50 kg.

3. The welding station can be modelled with a $M|M|2$-queue model and the painting station with a $M|M|1$-queue model. We can think of the bike shop as a Jackson network,



where $\lambda = 3$ bikes per hour, λ_A and $_B$ are the intensities at steady state to the welding and painting stations. They must satisfy $\lambda + 0.1\lambda_A = \lambda_A$ and $\lambda + 0.2\lambda_B = \lambda_B$ which yields, $\lambda_A = 10/3$ and $\lambda_B = 15/4$. We can now check that the low traffic requirements are satisfied, i.e., that $\lambda_A = 10/3 < 2 * \mu_A = 4$ and $\lambda_A = 15/4 < \mu_B = 4$, where μ_A and μ_B are the given service intensities of the workers are the stations.

(a) For a Jackson network the probabilities of a zero queue to the welding station and a zero queue to the painting station can be determined as the probability of having zero queues to independent queueing systems

$$P(N_A = 0)P(N_B = 0) + P(N_A = 0)P(N_B = 1) + P(N_A = 1)P(N_B = 0)$$

$$+P(N_A = 1)P(N_B = 1) + P(N_A = 2)P(N_B = 0) + P(N_A = 2)P(N_B = 1).$$

This is a lot of computations, so here we reduce the computations to show the probability of an empty system, no queues and no service,

$$P(N_A = 0, N_B = 0) = P(N_A = 0)P(N_B = 0) = \frac{1-\rho_A}{1+\rho_A}(1-\rho_B) =$$

$$= \frac{1-10/3/4}{1+10/3/4}(1-15/4/4) = 1/176$$

The average queue lengths are

$$L_A = \frac{2\rho_A}{1-\rho_A^2} = \frac{5}{3}\frac{36}{11}$$

and

$$L_B = \frac{\rho_B}{1-\rho_B} = \frac{5}{3}\frac{36}{11} = 15.$$

(b) Let V_A and V_B be the average time it takes from a car enters one of the service stations until it leaves it, i.e., $V_A = L_A/\lambda_A = 18/11$ and $V_B = L_B/\lambda_B = 4$.

Letting W_A be the average time from a frame arrives to station A until it leaves the factory and W_B be the average time from the frame arrives to station B until it leaves the factory, then

$$W_A = V_A + 0.1W_A + 0.9W_B$$

$$W_B = V_B + 0.2W_B$$

and $W_A = 675/99$ and $W_B = 4/0.8 = 5$.

The average time it takes for a bike to be welded and painted properly is approximately 6.7 hours.

4. (a) f and g are clearly seperable, so we can just check the properties of f_k and g_k for $k = 1, 2, 3$.

Note that $\Delta f_k(x) = -3$ so it is decreasing and $\Delta g_k(x) = k(x+1)^2$ so it is increasing.

Since $\Delta^2 f_k(x) = 0$ and $\Delta^2 g_k(x) = k(x+2)^2 - k(x+1)^2 > 0$ they are both integer convex (for positive x).

(b) When we apply the marginal allocation algorithm we want to compare the quotients $-\Delta f_k(x)/\Delta g_k(x)$ and find the largest elements when $k = 1, 2, 3$ and $x = 1, 2, 3, \cdots$. Here, $-\Delta f_k = 3$ is a constant so it is easier to find the smallest of the quotients $3\Delta g_k(x)/(-\Delta f_k(x)) = \Delta g_k(x)$.

n	$\Delta g_1(n)$	$\Delta g_2(n)$	$\Delta g_3(n)$
1	4	8	12
2	9	18	27
3	16	32	48
4	25	50	75

The smallest element is 4, so $n^{(4)} = (2, 1, 1)$ is the optimal allocation for sum of x is 4 and $f(n^{(4)}) = 48$, $g(n^{(4)}) = 10$.

The smallest element is 8, so $n^{(5)} = (2, 2, 1)$ is the optimal allocation for sum of x is 5 and $f(n^{(5)}) = 45$, $g(n^{(5)}) = 18$.

The smallest element is 9, so $n^{(6)} = (3, 2, 1)$ is the optimal allocation forsum of x is 6 and $f(n^{(6)}) = 42$, $g(n^{(6)}) = 27$

The smallest element is 12, so $n^{(7)} = (3, 2, 2)$ is the optimal allocation for sum of x is 7 and $f(n^{(7)}) = 39$, $g(n^{(7)}) = 39$

The smallest element is 16, so $n^{(8)} = (4, 2, 2)$ is the optimal allocation for sum of x is 8 and $f(n^{(8)}) = 36$, $g(n^{(8)}) = 55$

(c) The optimal soultion is $\hat{x} = (3, 2, 1)$ which corresponds to the efficient point with $g(\hat{x}) = 27$ and $f(\hat{x}) = 42$.

5. (a) We need to keep track of what the price of gas is and how much gas there is in the tank.

Define the state

$$x_k = \begin{cases} 1 & \text{if the price is high and the tank is full at beginning of day } k \\ 2 & \text{if the price is high and the tank is half-full at beginning of day } k \\ 3 & \text{if the price is low and the tank is full at beginning of day } k \\ 4 & \text{if the price is low and the tank is half-full at beginning of day } k \end{cases}$$

Define the decisions

$$c_k = \begin{cases} 1 & \text{if Frasse fills no gas at the end of day} k \\ 2 & \text{if Frasse fills half-tank of gas at the end of day } k \\ 3 & \text{if Frasse fills full-tank of gas at the end of day } k \end{cases}$$

Then the costs of making decision $c_k = 1$ is 0, the cost of making decision $c_k = 2$ is $H/2$ if $x_k = 1, 2$ and $L/2$ if $x_k = 3, 4$, and the cost of making decision $c_k = 3$ is $H/2$ if $x_k = 2$ and $L/2$ if $x_k = 4$.

In states $x_k = 1, 3$ the decision $c_k = 1, 2$ are feasible and in states $x_k = 2, 4$ the decision $c_k = 2, 3$ are feasible.

(b) It is best to buy as much gas as possible when the price is low and as little as possible when the price is high.

Guessed policy:
If $x_k = 1$, make decision $c_k = 1$, for the cost 0.
If $x_k = 2$, make decision $c_k = 2$, for the cost $H/2$.
If $x_k = 3$, make decision $c_k = 2$, for the cost $L/2$.
If $x_k = 4$, make decision $c_k = 3$, for the cost L.

This will give the probability transition matrix

$$P = \begin{bmatrix} 0 & 0.6 & 0 & 0.4 \\ 0 & 0.6 & 0 & 0.4 \\ 0.2 & 0 & 0.8 & 0 \\ 0.2 & 0 & 0.8 & 0 \end{bmatrix}$$

The stationary probabilities are given by the solution to $\pi = \pi P$ and $\sum \pi_i = 1$, i.e.

$$\pi = \frac{1}{15}(2, 3, 8, 2)$$

and the stationary cost is $1/15(0 \cdot 2 + H/2 \cdot 3 + L/2 \cdot 8 + L \cdot 2) = 3.2$.

(c) Use the policy iteration algorithm. Let $v_4 = 0$, then the value determination equations

$$g + v_1 = 0.6v_2$$

$$g + v_2 = H/2 + 0.6v_2$$

$$g + v_3 = L/2 + 0.2v_1 + 0.8v_3$$

$$g = L + 0.2v_1 + 0.8v_3$$

gives $v_1 = -2$, $v_2 = 2$, $v_3 = -3$ and $g = 3.2$ as in (b). (This calculation is an alternative solution method that can be used in b)

To find out if it is optimal we do one step of the policy iteration.

For $i = 1$

$$\min_{k=1,2}\{C_{1k} + (p_{11}(k)v_1 + p_{12}(k)v_2 + p_{13}(k)v_1 + p_{14}(k)v_4)\} =$$

$$= \min\{C_{11}+v(p_{11}(1)v_1+p_{12}(1)v_2+p_{13}(1)v_3+p_{14}(1)v_4), C_{12}+v(p_{11}(2)v_1+p_{12}(2)v_2+p_{13}(2)v_3+p_{14}(2)v_4)\}$$

$$= \min\{\underbrace{0 + (0v_1 + 0.6v_2 + 0v_3 + 0.4v_4)}_{1.2}, \underbrace{H/2 + (0.6v_1 + 0v_2 + 0.4v_3 + 0v_4)}_{1.6}\} = 1.2 = g+v_1 \text{ for } k = 1.$$

For $i = 2$

$$\min_{k=2,3}\{C_{2k} + (p_{21}(k)v_1 + p_{22}(k)v_2 + p_{23}(k)v_1 + p_{24}(k)v_4)\} =$$

$$= \min\{C_{22}+v(p_{21}(2)v_1+p_{22}(2)v_2+p_{23}(2)v_3+p_{24}(2)v_4), C_{23}+v(p_{21}(3)v_1+p_{22}(3)v_2+p_{23}(3)v_3+p_{24}(3)v_4)\}$$

$$= \min\{\underbrace{H/2 + (0v_1 + 0.6v_2 + 0v_3 + 0.4v_4)}_{5.2}, \underbrace{H + (0.6v_1 + 0v_2 + 0.4v_3 + 0v_4)}_{5.6}\} = 5.2 = g+v_2 \text{ for } k = 2.$$

For $i = 3$

$$\min_{k=1,2}\{C_{3k} + (p_{31}(k)v_1 + p_{32}(k)v_2 + p_{33}(k)v_1 + p_{34}(k)v_4)\} =$$

$$= \min\{C_{31}+v(p_{31}(1)v_1+p_{32}(1)v_2+p_{33}(1)v_3+p_{34}(1)v_4), C_{32}+v(p_{31}(2)v_1+p_{32}(2)v_2+p_{33}(2)v_3+p_{34}(2)v_4)\}$$

$$= \min\{\underbrace{0 + (0v_1 + 0.2v_2 + 0v_3 + 0.8v_4)}_{0.4}, \underbrace{L/2 + (0.2v_1 + 0v_2 + 0.8v_3 + 0v_4)}_{0.2}\} = 0.2 = g+v_3 \text{ for } k = 2.$$

For $i = 4$

$$\min_{k=2,3}\{C_{4k} + (p_{41}(k)v_1 + p_{42}(k)v_2 + p_{43}(k)v_1 + p_{44}(k)v_4)\} =$$

$$= \min\{C_{42}+v(p_{41}(2)v_1+p_{42}(2)v_2+p_{43}(2)v_3+p_{44}(2)v_4), C_{43}+v(p_{41}(3)v_1+p_{42}(3)v_2+p_{43}(3)v_3+p_{44}(3)v_4)\}$$

$$= \min\{\underbrace{L/2 + (0v_1 + 0.2v_2 + 0v_3 + 0.8v_4)}_{3.4}, \underbrace{L + (0.2v_1 + 0v_2 + 0.8v_3 + 0v_4)}_{3.2}\} = 3.2 = g+v_4 \text{ for } k = 3.$$

So the guessed strategy is optimal.