KTH Mathematics

Suggested solutions for the exam in SF2863 Systems Engineering.
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1. We can think of the support center as a Jackson network. The reception is a $M|M|2$ queue with arrival intensity λ_R and service intensity $\mu_R = 24$ per server. Frasse's service is a $M|M|1$ queue with arrival intensity λ_F and service intensity $\mu_F = 40$ per server. The reception is a $M|M|1$ queue with arrival intensity λ_H and service intensity $\mu_R = 20$ per server.

(a) Let $\lambda_R = 40$ be the arrival intensity from the outside, i.e. the intensity in to the reception. The traffic balance equations are $\lambda_F = 3/4\lambda_R + 1/2\lambda_H$ and $\lambda_H = 1/4\lambda_R + 1/5\lambda_F$ which yields, $\lambda_F = 350/9$ and $\lambda_H = 160/9$. We can now check that the low traffic requirements are satisfied, i.e., that $\lambda_R = 40 < 2 * \mu_R = 48$, $\lambda_F = 350/9 < \mu_F = 40$, and $\lambda_H = 160/9 < \mu_H = 20$.

$L_R = 2\rho_R/(1-\rho_R^2) = 60/11$, $L_F = \rho_F/(1-\rho_F) = 35$, and $L_H = \rho_H/(1-\rho_H) = 8$.

The average total number of customers in the systems is then $L = L_R + L_H + L_F = 43 + 60/11 \approx 48.5$

(b) Let V_R, V_F and V_H be the average time it takes from a call arrives to one of the service stations until it leaves it, i.e., $V_R = L_R/\lambda_R = 2\rho_R/(1-\rho_R^2)/\lambda_R = 60/11/40 = 3/22$, $V_F = L_F/\lambda_F = \rho_F/(1-\rho_F)/\lambda_F = 35/(350/9) = 9/10$, and $V_H = L_H/\lambda_H = \rho_H/(1-\rho_H)/\lambda_H = 8/(160/9) = 9/20$.

Letting W_R be the average time from a call arrives to station R until it exits the system , W_F be the average time from a call arrives to station F until it exits the system, and W_H be the average time from the call arrives to station H until it exits the system, then

$$W_R = V_R + 3/4W_F + 1/4W_H$$

$$W_F = V_F + 1/5W_H$$

$$W_H = V_H + 1/2W_F$$

and $W_R = 1.21$.

Alternatively, Wilsons formula says that $W_R = L/\Lambda_R = 48.5/40 = 1.21$.

The average cost is then $W_R * 4 * 60 = 291$ SEK

2. This can be solved using the newsboy problem formulation. Let $p = m = 20$, $h = k - g = 40 - 10 = 30$ and $c = C - k = 30 - 40 = -10$.

(a) Then the optimal target price satisfies $\frac{p-c}{p+h} = \frac{20+10}{20+30} = \frac{30}{50} = F(S^*)$.
F is the cumulative distribution function of the demand, which is assumed to be uniform on the interval (0,100), i.e. $F(t) = t/100$ on the interval. $F(60) = 30/50$ and therefore $S^* = 60$.

(b) The forward difference of the expected cost is

$$\Delta C(59) = C(60) - C(59) = -10 + (20)[60/101 - 1] + (30)60/101 = \frac{-1010 - 820 + 1800}{101} = -\frac{30}{101}$$

and

$$\Delta C(60) = C(61) - C(60) = -10 + (20)[61/101 - 1] + (30)61/101 = \frac{-1010 - 800 + 1830}{101} = \frac{20}{101}$$

so C decreases from 59 to 60 and then increases from 60 to 61. Therefore, there is a local minimum at $C = 60$, which is also a global minimum since $\Delta^2 C(S) = (p+h)\Delta F_\xi(S) = 50P(\xi = S+1) \geq 0$ shows that the expected cost is an integer-convex function.

3. (a) Let $x_i = 1$ if app nr i is on the startpage and $x_i = 0$ if app nr i is not on the startpage.
Let $f(x_1, \cdots, x_N) = -\sum_{i=1}^N x_i v_i$ and $g(x_1, \cdots, x_N) = \sum_{i=1}^N x_i s_i$.
Then f and g are separable functions, f is decreasing and g is increasing. Furthermore, $\Delta^2 f = \Delta^2 g = 0$ since the functions are linear, so they are both integer-convex. We can then apply the Marginal Allocation algorithm.
Note that $\Delta f_i(x) = -v_i$ and $\Delta g_i(x) = v_i$, so the quotients $-\Delta F_i(x)/\Delta g_i(x) = v_i/s_i$ does not depend on x. Here $v_1/s_1 = 1/1$, $v_2/s_2 = 2/1$, $v_3/s_3 = 5/2$, $v_4/s_4 = 6/2$, $v_5/s_5 = 8/3$.
The efficient allocations are therefore,
$x^{(0)} = (x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0, x_5 = 0)$, $\mathrm{f}(\mathrm{x}^{(0)}) = 0, g(x^{(0)}) = 0$
$\mathrm{x}^{(1)} = (x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 1, x_5 = 0)$, $\mathrm{f}(\mathrm{x}^{(1)}) = -6, g(x^{(1)}) = 2$
$\mathrm{x}^{(2)} = (x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 1, x_5 = 1)$, $\mathrm{f}(\mathrm{x}^{(2)}) = -14, g(x^{(2)}) = 5$
$\mathrm{x}^{(3)} = (x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1, x_5 = 1)$, $\mathrm{f}(\mathrm{x}^{(3)}) = -19, g(x^{(3)}) = 7$
$\mathrm{x}^{(4)} = (x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 1, x_5 = 1)$, $\mathrm{f}(\mathrm{x}^{(4)}) = -21, g(x^{(4)}) = 8$
$\mathrm{x}^{(5)} = (x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1, x_5 = 1)$, $\mathrm{f}(\mathrm{x}^{(5)}) = -22, g(x^{(5)}) = 9$

(b) The optimization problem is then

$$\begin{array}{rl} \max\limits_{x_1,\cdots,x_M} & \sum_{i=1}^M x_i v_i \\ \text{s.t.} & \sum_{i=1}^M x_i s_i \leq N \\ & x_i \in \{0,1\} \text{ for } i = 1, \cdots, M. \end{array}$$

(c) Introduce the stage k as the reduced problem when Frasse has only apps of type $k, \cdots, M$ to choose from. Introduce the state, n_k = how many free spaces Frasse has at stage k, which we assume is non-negative.
Let

$$(\mathcal{P}_k(n)) \qquad \begin{array}{rl} \max\limits_{x_k,\cdots,x_M} & \sum_{i=k}^M x_i v_i \\ \text{s.t.} & \sum_{i=k}^M x_i s_i \leq n \\ & x_i \in \{0,1\} \text{ for } i = k, \cdots, M. \end{array}$$

and define $f_k^*(n)$ to be the function mapping n to the optimal value of $(\mathcal{P}_k(n))$.
Then $n_{k-1} = n_k - s_k x_k$.
The DynP recursion can be written as

$$f_\ell^*(n_\ell) = \max_{x_\ell=0,1} \left\{ v_\ell x_\ell + f_{\ell+1}^*(n_\ell - s_\ell x_\ell) \right\} = \max \left\{ f_{\ell+1}^*(n_\ell), v_\ell + f_{\ell+1}^*(n_\ell - s_\ell) \right\},$$

and the boundary condition is that $f_{M+1}^* = 0$.
The DynP problem is solved in the attached figure, where $f_1^*(0) = 0$, $f_1^*(1) = 2$, $f_1^*(2) = 6$, $f_1^*(3) = 8$, $f_1^*(4) = 11$, $f_1^*(5) = 14$, $f_1^*(6) = 16$, $f_1^*(7) = 19$, $f_1^*(8) = 21$.
The $\hat{x} = (0, 1, 1, 1, 1)$ is optimal for $N = 8$.

4. (a) We need to keep track of the position of the pawn, define the state

$$s_k = \begin{cases} 0 & \text{if pawn is on square 0} \\ 1 & \text{if pawn is on square 1} \\ 2 & \text{if pawn is on square 2} \\ 3 & \text{if pawn is on square 3} \end{cases}$$

Define the decisions

$$x_k = \begin{cases} 1 & \text{if Frasse uses die 1 turn } k \\ 2 & \text{if Frasse uses die 2 turn } k \end{cases}$$

The transition probabilities are
$p_{ij}(k) =$ *the probability of jumping from state i to j if we make decision k.*
Here, from square 0
$p_{00}(1) = 0$, $p_{01}(1) = 1/2$, $p_{02}(1) = 1/3$, $p_{03}(1) = 1/6$,
$p_{00}(2) = 0$, $p_{01}(2) = 1/3$, $p_{02}(2) = 1/3$, $p_{03}(2) = 1/3$,

Here, from square 1
$p_{10}(1) = 1/6$, $p_{11}(1) = 0$, $p_{12}(1) = 1/2$, $p_{13}(1) = 1/3$,
$p_{10}(2) = 1/3$, $p_{11}(2) = 0$, $p_{12}(2) = 1/3$, $p_{13}(2) = 1/3$,

Here, from square 2
$p_{20}(1) = 1/2$, $p_{21}(1) = 0$, $p_{22}(1) = 0$, $p_{23}(1) = 1/2$,
$2p_{20}(2) = 2/3$, $p_{21}(2) = 0$, $p_{22}(2) = 0$, $p_{23}(2) = 1/3$,

Here, from square 3
$p_{30}(1) = 1$, $p_{31}(1) = 0$, $p_{32}(1) = 0$, $p_{33}(1) = 0$,
$p_{30}(2) = 1$, $p_{31}(2) = 0$, $p_{32}(2) = 0$, $p_{33}(2) = 0$,

Then the expected "cost" of making decision x_k at state s_k is $C_{s_k,x_k} = \sum_{j=0}^{3} q_{s_k,j} p_{s_k,j}(x_k)$.
(The cost is an income here, so we maximize it instead)
So at $s_k = 0$
$C_{01} = 0 \cdot 0 + 1 \cdot 1/2 + 2 \cdot 1/3 + 3 \cdot 1/6 = 5/3$ for $x_k = 1$
$C_{02} = 0 \cdot 0 + 1 \cdot 1/3 + 2 \cdot 1/3 + 3 \cdot 1/3 = 2$ for $x_k = 2$

So at $s_k = 1$
$C_{11} = 0 \cdot 1/6 + 1 \cdot 0 + 2 \cdot 1/2 + 3 \cdot 1/3 = 2$ for $x_k = 1$
$C_{12} = 0 \cdot 1/3 + 1 \cdot 0 + 2 \cdot 1/3 + 3 \cdot 1/3 = 5/3$ for $x_k = 2$

So at $s_k = 2$
$C_{21} = 0 \cdot 1/2 + 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 1/2 = 3/2$ for $x_k = 1$
$C_{22} = 0 \cdot 2/3 + 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 1/3 = 1$ for $x_k = 2$

So at $s_k = 3$
$C_{31} = 0 \cdot 1 + 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 = 0$ for $x_k = 1$
$C_{32} = 0 \cdot 1 + 1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 = 0$ for $x_k = 2$

(b) Let y_{ik} = probability of beeing in state i and making decision k.
Let the objective function be defined by (if we maximize, otherwise we have to put a minus in front).

$$f = C_{01}y_{01} + C_{11}y_{11} + C_{21}y_{21} + C_{31}y_{31} + C_{02}y_{02} + C_{12}y_{12} + C_{22}y_{22} + C_{32}y_{32}$$

One constraint is (summing all probabilities)

$$y_{01} + y_{11} + y_{21} + y_{31} + y_{02} + y_{12} + y_{22} + y_{32} = 1$$

non-negativity constraints

$$y_{01} \geq 0, y_{11} \geq 0, y_{21} \geq 0, y_{31} \geq 0, y_{02} \geq 0, y_{12} \geq 0, y_{22} \geq 0, y_{32} \geq 0.$$

finally

$$y_{01}+y_{02}-(p_{00}(1)y_{01} + p_{10}(1)y_{11} + p_{20}(1)y_{21} + p_{30}(1)y_{31} + p_{00}(2)y_{02} + p_{10}(2)y_{12} + p_{20}(2)y_{22} + p_{30}(2)y_{32}) = 0$$

$$y_{11}+y_{12}-(p_{01}(1)y_{01} + p_{11}(1)y_{11} + p_{21}(1)y_{21} + p_{31}(1)y_{31} + p_{01}(2)y_{02} + p_{11}(2)y_{12} + p_{21}(2)y_{22} + p_{31}(2)y_{32}) = 0$$

$$y_{01}+y_{02}-(p_{02}(1)y_{01} + p_{12}(1)y_{11} + p_{22}(1)y_{21} + p_{32}(1)y_{31} + p_{02}(2)y_{02} + p_{12}(2)y_{12} + p_{22}(2)y_{22} + p_{32}(2)y_{32}) = 0$$

$$y_{01}+y_{02}-(p_{03}(1)y_{01} + p_{13}(1)y_{11} + p_{23}(1)y_{21} + p_{33}(1)y_{31} + p_{03}(2)y_{02} + p_{13}(2)y_{12} + p_{23}(2)y_{22} + p_{33}(2)y_{32}) = 0$$

One of these last four should be removed due to linear dependence.
Then, from the solution of the linear program we can determine $\pi_0 = y_{01} + y_{02}$,$\pi_1 = y_{11} + y_{12}$,$\pi_2 = y_{21} + y_{22}$ and $\pi_3 = y_{31} + y_{32}$ which are the stationary probabilities of beeing in a certain state..
Finally $D_{ik} = y_{ik}/\pi_i$, determines the optimal policy. D_{ik} is defined as the probability of making decision k in state i, but it can be shown that it will be either 0 or 1, i.e., a deterministic policy.

(c) Starting policy:
If $s_k = 0$, make decision $x_k = 2$, for the expected cost 2.
If $s_k = 1$, make decision $x_k = 1$, for the expected cost 2.
If $s_k = 2$, make decision $x_k = 1$, for the expected cost 3/2.

If $s_k = 3$, make decision $x_k = 1$, for the expected cost 0.

Use the policy iteration algorithm. Let $v_3 = 0$, then the value determination equations

$$g + v_0 = 2 + 0v_0 + 1/3v_1 + 1/3v_2 + 1/3v_3$$
$$g + v_1 = 2 + 1/6v_0 + 0v_1 + 1/2v_2 + 1/3v_3$$
$$g + v_2 = 3/2 + 1/2v_0 + 0v_1 + 0v_2 + 1/2v_3$$
$$g + v_3 = 0 + 1v_0 + 0v_1 + 0v_2 + 0v_3$$

gives $g = 123/91$, $v_0 = 123/91$, $v_1 = 9/7$, $v_2 = 75/91$ and $v_3 = 0$.
We see that v_0 is the largest of the v_i, so it is best to start from square 0 if we can choose. (assuming we use the initial policy)
To find out if it is optimal we do one step of the policy iteration.
For $i = 0$

$$\max_{k=1,2}\{C_{0k} + (p_{00}(k)v_0 + p_{01}(k)v_1 + p_{02}(k)v_2 + p_{03}(k)v_3)\} =$$

$$= \max\{C_{01}+(p_{00}(1)v_0+p_{01}(1)v_1+p_{02}(1)v_2+p_{03}(1)v_3), C_{02}+(p_{00}(2)v_0+p_{01}(2)v_1+p_{02}(2)v_2+p_{03}(2)v_3)\}$$
$$= \max\{\underbrace{5/3 + (1/2v_1 + 1/3v_2)}_{1411/546}, \underbrace{2 + (1/3v_1 + 1/3v_2)}_{246/91}, \} = 246/91 \text{ for } k = 2.$$

For $i = 1$

$$\max_{k=1,2}\{C_{1k} + (p_{10}(k)v_0 + p_{11}(k)v_1 + p_{12}(k)v_2 + p_{13}(k)v_3)\} =$$

$$= \max\{C_{11}+(p_{10}(1)v_0+p_{11}(1)v_1+p_{12}(1)v_2+p_{13}(1)v_3), C_{12}+(p_{10}(2)v_0+p_{11}(2)v_1+p_{12}(2)v_2+p_{13}(2)v_3)\}$$
$$= \max\{\underbrace{2 + (1/6v_0 + 1/2v_2)}_{240/91}, \underbrace{5/3 + (1/3v_0 + 1/3v_2)}_{653/273}, \} = 240/91 \text{ for } k = 1.$$

For $i = 2$

$$\max_{k=1,2}\{C_{2k} + (p_{20}(k)v_0 + p_{21}(k)v_1 + p_{22}(k)v_2 + p_{23}(k)v_3)\} =$$

$$= \max\{C_{21}+(p_{20}(1)v_0+p_{21}(1)v_1+p_{22}(1)v_2+p_{23}(1)v_3), C_{22}+(p_{20}(2)v_0+p_{21}(2)v_1+p_{22}(2)v_2+p_{23}(2)v_3)\}$$
$$= \max\{\underbrace{3/2 + (1/2v_0 + 1/2v_2)}_{198/91}, \underbrace{1 + 2/3v_0}_{173/91}, \} = 198/91 \text{ for } k = 1.$$

For $i = 3$

$$\max_{k=1,2}\{C_{3k} + (p_{30}(k)v_0 + p_{31}(k)v_1 + p_{32}(k)v_2 + p_{33}(k)v_3)\} =$$

$$= \max\{C_{31}+(p_{30}(1)v_0+p_{31}(1)v_1+p_{32}(1)v_2+p_{33}(1)v_3), C_{32}+(p_{30}(2)v_0+p_{31}(2)v_1+p_{32}(2)v_2+p_{33}(2)v_3)\}$$
$$= \max\{\underbrace{0 + (1v_0)}_{123/91}, \underbrace{0 + (1v_0)}_{123/91}, \} = 123/91 \text{ for } k = 1. \text{ (or 2)}$$

So the initial policy is optimal.