KTH Mathematics

Suggested solutions for the exam in SF2863 Systems Engineering.
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1. The intensity of arrivals from outside to Goran is $\lambda/2$, and the arrivals to his queue λ_G then satisfies

$$\lambda/2 + 0.2\lambda_G = \lambda_G,$$

i.e., $\lambda_G = 5/8\lambda$. The intensity for his service is $\mu_G = 3$ exams per hour, then $\rho_G = \lambda_G/\mu_G = 5/24\lambda$, and symmetrically $\lambda_H = 5/8\lambda$, $\mu_H = 3$ and $\rho_H = 5/24\lambda$.

Now, at steady state, the number of exams $160 = L_G + L_H = \frac{\rho_G}{1-\rho_G} + \frac{\rho_H}{1-\rho_H}$, which say that $\lambda = 128/27$ [exams per hour].

The service intensity for Pedro is $\mu_P = 12$ exams per hour. The arrival intensity to Pedros desk is $\lambda_P = 0.8\lambda_G + 0.8\lambda_P = 2\frac{4}{5}\frac{5}{8}\frac{384}{81} = 128/27 = \lambda$, and average number of exams handled by Pedro is $L_P = \frac{(128/27)/12}{1-(128/27)/12} = 32/49$

The total number of exams is then $L_{tot} = L_G + L_H + L_P = 160 + 32/49$.

The expected time to pass through the system is $W_{tot} = L_{tot}/\lambda = (160{+}32/103)/(384/81) = 3483/103$ or approximately 34 hours.

The probability that an $M|M|1$ system is empty is $P_0 = 1 - \rho$.

For Goran and Hilda the proportion of time idling is $1 - 80/81 = 1/81$, for Pedro it is $1 - 128/27/12 \approx 0.65$.

The proportion of time that they are all idling at the same time is, since we can calculate as if independent, $49/81^2 \cdot 0.65 \approx 1e-4$.

2. Define the variables.

Let s_n = number of students in class n.

Let the decision $x_n = 1$ if Goran gives ordinary class at class n,
and the decision $x_n = 2$ if Goran offers cake at class n,
and the decision $x_n = 3$ if Goran offers hint at class n.

Let $V_n(s_n)$ =optimal total number of students to goal if at start of class n he has s_n students.

Let $V_n(s_n, x)$ =optimal total number of students to goal if at start of class n he has s_n students and decision x is taken about class n.

If $x_k = 1$, then the number of students at next class is $s_{k+1} = 0.9s_k$.
If $x_k = 2$, then the number of students at next class is $s_{k+1} = s_k + 10$.
If $x_k = 3$, then the number of students at next class is $s_{k+1} = s_k + (100 - s_k)/2 = 50 + s_k/2$.

Let $c_1 = 0$, $c_2 = 15$ and $c_3 = 40$ be the immediate costs for using the different decisions.

Then

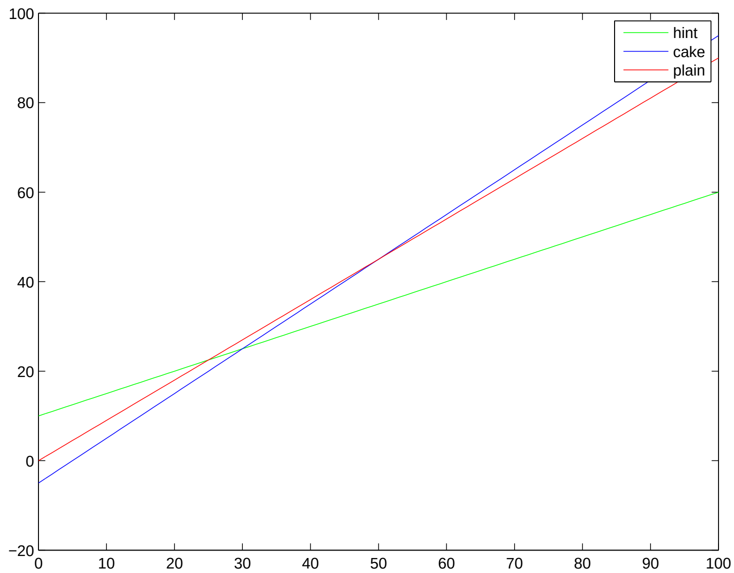
$$V_n(s) = \max_{x\in\{1,2,3\}} V_n(s,x) = \min\left\{s - c_1 + V_{n+1}(0.9s), s - c_2 + V_{n+1}(s+10), s - c_3 + V_{n+1}(50 + s/2)\right\},$$

and it is easy to see that at the start of the last class, class 3, the best is to give an ordinary class, so $V_3(s) = s$.

Then

$$V_2(s) = \max_{x\in\{1,2,3\}} V_2(s,x) = \min\left\{s + 0.9s, s - 15 + s + 10, s - 40 + 50 + s/2\right\},$$

From the Figure, illustrating $V_2(s,x) - s$,



it is clear that

$$V_2(s) = \begin{cases} s + 10 + s/2 & 0 \le s \le 25 & \hat{x} = 3,\ \text{hint} \\ s + 0.9s & 25 \le s \le 50 & \hat{x} = 1,\ \text{ordin} \\ 2s - 5 & 50 \le s \le 25 & \hat{x} = 2,\ \text{cake} \end{cases}$$

Finally, we should determine $V_1(40)$.

$$V_1(40) = \max_{x\in\{1,2,3\}} V_2(s,x) = \max\left\{40 + V_2(36), 40 - 15 + V_2(50), 40 - 40 + V_2(50 + 20)\right\},$$

$$= \max\left\{40 + 36 + 0.9 \cdot 36, 25 + 95, 2 \cdot 70 - 5\right\} = 135.$$

So the optimal value is 135 “students”.

The first class he should offer a hint. The second class he should offer cake. The last class it is always optimal to give an ordinary class.

3. Let s_i denote the number of days dedicated to studying exam i, $i = 1, 2, 3$..

Define functions f and g that we want to minimize, with the right properties. Maximizing the expected number of passed exams is the same as minimizing minus the sum of the probabilities of passing the exams, $p_i(s_i) = 1 - (s_i - A_i)^2/B_i$.

Let $f(d_1, d_2, d_3, d_4) = -[p_1(s_1) + p_2(s_2) + p_3(s_3)]$ and $g(s_1, s_2, s_3) = \sum_{i=1}^{3} s_i$. Clearly g is a separable function, increasing and integer-convex.

The continuous version of function f has a gradient $\nabla f = (2(s_1 - 10)/100, 2(s_2 - 8)/100, 2(s_3 - 12)/150)$ which has negative elements for s_i less than 8, which makes the function decreasing for the range we consider. Furthermore, f is seperable, $f = f_1(s_1) + f_2(s_2) + f_3(s_3)$, and the functions f_i are quadratic functions which are convex, since the Hessian is diagonal with positive elements..

(Note that $\Delta f_i(x) \neq f_i'(x)$)

$-\Delta f$	$-\Delta f_1$	$-\Delta f_2$	$-\Delta f_3$
$s_i = 1$	$19/100$	$15/100$	$23/150$
$s_i = 2$	$17/100$	$13/100$	$21/150$
$s_i = 3$	$15/100$	$11/100$	$19/150$
$s_i = 4$	$13/100$	$9/100$	$17/150$
$s_i = 5$	$13/100$	$7/100$	$15/150$

Since the values are decreasing in each column the function f is integer-convex for the tabulated values.

Note that $\Delta g_i = 1$ for all i.

We can then apply the Marginal Allocation algorithm.

Note that the quotients $-\Delta f_i(s)/\Delta g_i(s) = \Delta f_i(s_i)$ are given in the table above.

The efficient allocations are therefore,
$S^{(0)} = (s_1 = 0, s_2 = 0, s_3 = 0),\ f(s^{(0)}) = -10/25,\ g(s^{(0)}) = 0$
$S^{(1)} = (s_1 = 1, s_2 = 0, s_3 = 0),\ f(s^{(1)}) = -59/100,\ g(s^{(1)}) = 1$
$S^{(2)} = (s_1 = 2, s_2 = 0, s_3 = 0),\ f(s^{(2)}) = -76/100,\ g(s^{(2)}) = 2$
$S^{(3)} = (s_1 = 2, s_2 = 0, s_3 = 1), f(s^{(3)}) = -137/150,\ g(s^{(3)}) = 3$
$S^{(4)} = (s_1 = 3, s_2 = 0, s_3 = 1),\ f(s^{(4)}) = -319/300,\ g(s^{(4)}) = 4$
$S^{(5)} = (s_1 = 3, s_2 = 1, s_3 = 1), f(s^{(5)}) = -91/75,\ g(s^{(5)}) = 5$

For $s^{(4)}$ there is a an alternative solution with $s_1 = 2$ and $s_2 = 1$.

The expected number of passed exams is $-f(s^{(k)})$.

4. We need to keep track of if Clyde passed or failed the last exam. Let the state be

$$s_k = \begin{cases} 1 & \text{if Clyde passed last exam, at time } k-1 \\ 0 & \text{if Clyde failed last exam, at time } k-1 \end{cases}$$

Define the decisions

$$x_k = \begin{cases} 1 & \text{if Frasse decides to study for next exam, at time } k \\ 0 & \text{if Frasse decides to not study for next exam, at time } k \end{cases}$$

The transition probabilities are
$p_{ij}(x) =$ *the probability of jumping from state i to j if we make decision x.*

$$P(x=1) = \begin{bmatrix} p_{11}(1) & p_{10}(1) \\ p_{01}(1) & p_{00}(1) \end{bmatrix} = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

$$P(x=0) = \begin{bmatrix} p_{11}(0) & p_{10}(0) \\ p_{01}(0) & p_{00}(0) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/8 & 7/8 \end{bmatrix}$$

Let the costs be the utility and maximize instead of minimize.

Let $q_{ij}(x) =$ expected utility incurred when the state is i decision x is made and the system evolves to state j.

$$Q(x=1) = \begin{bmatrix} q_{11}(1) & q_{10}(1) \\ q_{01}(1) & q_{00}(1) \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 10 & 0 \end{bmatrix}$$

$$Q(x=0) = \begin{bmatrix} q_{11}(0) & p_{10}(0) \\ q_{01}(0) & p_{00}(0) \end{bmatrix} = \begin{bmatrix} 38 & 6 \\ 20 & 4 \end{bmatrix}$$

Then the expected "cost" of making decision x_k at state s_k is $C_{s_k,x_k} = \sum_{j=0}^{3} q_{s_k,j} p_{s_k,j}(x_k)$.

$$C_{00} = q_{00}(0)p_{00}(0) + q_{01}(0)p_{01}(0) = 6$$

$$C_{01} = q_{00}(1)p_{00}(1) + q_{01}(1)p_{01}(1) = 2.5$$

$$C_{10} = q_{10}(0)p_{10}(0) + q_{11}(0)p_{11}(0) = 22$$

$$C_{11} = q_{10}(1)p_{10}(1) + q_{11}(1)p_{11}(1) = 8$$

Starting policy:
If $s_k = 0$, make decision $x_k = 1$.
If $s_k = 1$, make decision $x_k = 1$.

Use the policy iteration algorithm. Let $v_1 = 0$, then the value determination equations

$$g + v_0 = 2.5 + (3/4)v_0 + (1/4)v_1$$

$$g + v_1 = 8 + (1/4)v_0 + (3/4)v_1$$

gives $g = 21/4$, $v_0 = -11$ and $v_1 = 0$.

To find out if it is optimal we do one step of the policy iteration.

For $i = 0$

$$\max_{k=0,1}\{C_{0k} + (p_{00}(k)v_0 + p_{01}(k)v_1)\} =$$

$$= \max\{C_{00} + (p_{00}(0)v_0 + p_{01}(0)v_1), C_{01} + (p_{00}(1)v_0 + p_{01}(1)v_1)\}$$

$$= \max\{\underbrace{6 + 7/8 * (-11)}_{-29/8}, \underbrace{2.5 + 3/4 * (-11)}_{-23/4}\} = -29/8 \text{ for } k = 0.$$

For $i = 1$

$$\min_{k=0,1}\{C_{1k} + (p_{10}(k)v_0 + p_{11}(k)v_1)\} =$$

$$= \max\{C_{10} + (p_{10}(0)v_0 + p_{11}(0)v_1), C_{11} + (p_{10}(1)v_0 + p_{11}(1)v_1)\}$$
$$= \max\{\underbrace{22 + 1/2 * (-11)}_{33/2}, \underbrace{8 + 1/4 * (-11)}_{21/4}, \} = 33/2 \text{ for } k = 0.$$

The starting policy is not optimal, to maximize the utility in the long run it is better to party all the time, if you trust Frasse....

5. (a) This can be modelled as an economic order quantity model, EOQ model. The demand is $d = 1500$ packages of noodles per day. The holding cost is $h = 0.03$ SEK per package and day. The ordering cost is $K = 1000$ SEK. Then the quantity to be ordered is

$$\hat{Q} = \sqrt{\frac{2dK}{h}} = 10000 \text{ packages.}$$

If the order takes two days to arrive, the order should be made two days before the inventory level reaches zero, *i.e.*, it should be ordered when the inventory level is at 3000 packages, and he should order every 10000/1500 day.

(b) This is a an EOQ model where shortages are allowed. The shortage cost is $p = 0.1$ SEK per day and per package of shortage. The total cost per time unit is determined in the book at page 837,

$$T = \frac{dK}{Q} + dc + \frac{hS^2}{2Q} + \frac{p(Q-S)^2}{2Q}$$

and the equation system is given by

$$T'_S = \frac{hS}{Q} - \frac{p(Q-S)}{Q} = 0$$

$$T'_Q = -\frac{dK}{Q^2} - \frac{hS^2}{2Q^2} + \frac{p(Q-S)}{Q} - \frac{p(Q-S)^2}{2Q^2} = 0$$

and the optimal Q and S are given by

$$Q^* = \sqrt{\frac{2dK}{h}}\sqrt{\frac{p+h}{p}} = 11402 \quad S^* = \sqrt{\frac{2dK}{h}}\sqrt{\frac{p}{p+h}} = 8771$$

The solution from (a) with no shortage is a feasible solution to the problem with planned shortage, so by the optimality of the solution in (b) it must be at least as good as the one in (a).